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HOMOGENEOUS KOSZUL MANIFOLDS IN $\mathbf{C}^n$

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0. Introduction

Let X be a complex manifold. We shall say that X is homogeneous under the real analytic Lie group G if X is a homogeneous G-space for which the mapping $\nu: G \times X \to X$ is real analytic in the G variable and holomorphic in the X variable. Suppose that there is a G-invariant volume form ω on X. In local coordinates, we may express ω as

$$\sqrt{-1}/2K(z, \overline{z})\, z_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n .$$

Homogeneity implies that K is strictly positive. In [6], Koszul introduced the following Hermitian form which we refer to as the Koszul form:

$$H = \sum \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K \, dz_i \, d\overline{z}_j . \tag{1}$$

The form H is invariant under any biholomorphic mapping which preserves the volume form. We shall say that X is a Koszul manifold if H is nondegenerate. This gives X the structure of a pseudo-Kähler manifold for which the measure preserving biholomorphisms are isometries.

In [6], Koszul proposed the problem of the classifying all Koszul manifolds. Considerable progress has been made on this problem in the cases when the Koszul structure is in fact Kähler (see [3] and the references contained therein) and in the symmetric case. In the general pseudo-Kähler case, it seems that very little progress has been made. In this work, we begin the study of this problem. We restrict to the case that G contains an exponential solvable group which acts transitively on X. We refer to such manifolds as "type E." (Note that all Hermitian symmetric spaces are type E. Also, all rationally homogeneous, contractible domains are type E.)

Our first main result includes the statement that all type E manifolds have realizations as homogeneous domains in $\mathbf{C}^n$. The result, however,

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is much more explicit. In [11] we introduced a class of domains referred to as Siegel N-P domains. In this notation, N is a connected, simply connected, nilpotent Lie group and P is a complex subgroup of the complexification N_c of N. We let A_p be the group of automorphisms of N which fix P, when extended holomorphically to N_c. Choose a maximal **R**-split torus $T \subset A_p$. The pair N-P is a Siegel pair if T has an open orbit in $N \backslash N_c/P$. (This is independent of the torus T.) In this case, the semidirect product $T \times_s N$ has a finite number of open orbits in N_c/P. The latter space is biholomorphic to $\mathbf{C}^n$ and each of the open orbits is a homogeneous domain which is referred to as a Siegel N-P domain. Note that N-P domains are transitive under a *completely solvable* group. In particular, they are type E. Our first main result is Theorem 1 below. The proof uses results of Hano [5]. (We are deeply indebted to S. Dorfmeister for bringing these results to our attention.) We remind the reader that a semialgebraic group is, by definition, the identity component (in the Euclidean topology) of the set of real points of a real algebraic group.

Theorem 1. *X may be realized as a Siegel N-P domain in $\mathbf{C}^n$. The identity component G_ω of the holomorphic isometry group of X is a semialgebraic group with trivial center for which the mapping $G_\omega \times X \to X$ is rational in both variables and is holomorphic in the X variable.*

One corollary of this result will be of special importance to us. Let $S \subset G$ be a connected, solvable subgroup of a real Lie group G. The group S is said to be triangular in G if there is **R**-basis of $\mathscr{G}$ under which the image of S under the adjoint representation is given by upper triangular matrices. It is a fundamental result of Mostow that all maximal connected triangular in G subgroups are conjugate under an inner automorphism.

The following is a consequence of Theorem 1. We remind the reader that a Lie group G is said to be ad-algebraic if its image under the adjoint map is semialgebraic.

Theorem 2. *Suppose that G is* ad-*algebraic and that X is type* E. *Then there is a triangular subgroup S of G which acts transitively on X.*

Using Theorem 2, we define a "rank" for a general type E manifold X. We then restrict to type E, rank one case. Such manifolds are called type I. In our work [11], we introduced a special class of domains, referred to as nil-balls. (We shall define this class, as well as the "dilated" nil-balls in §2 below.) These domains, it turns out, are just the type I manifolds. We prove

Theorem 3. *A manifold is type* I *if and only if it is biholomorphic to a homogeneous nil-ball.*

Theorem 3 contains a great deal of information. In particular, we may immediately draw the conclusions below.

Corollary 4. *Every type* I *manifold has a realization as a domain* Ω *in* $\mathbf{C}^n$ *such that*:

(1) *there is a solvable, algebraic group* G *with a codimension-one nilradical* N *which acts homogeneously and polynomially on* Ω,

(2) *the boundary* $\partial\Omega$ *is a smooth, algebraic, real codimension-one submanifold of* $\mathbf{C}^n$, *transitive under* N,

(3) *the Levi form is nondegenerate at each point of* $\partial\Omega$,

(4) *the complement of* $\overline{\Omega}$ *is homogeneous under* G.

We refer to this realization as the nil-ball realization. An example of such a realization would be the realization of the unit ball in $\mathbf{C}^n$ as a Siegel domain. The condition described in Theorem 1 above is referred to as "birational homogeneity." If we only assume rationality in the Ω variable (as in (1) below), then the action is referred to as "rationally homogenous." It is one of the main results of [11] that the existence of a rationally homogeneous action implies the existence of a birationally homogeneous action.

We conjecture that a contractable, homogeneous domain with a analytic, Levi nondegenerate boundary must be a type I domain. We have not been able to prove this in general. However, let Ω be a contractable domain in $\mathbf{C}^n$ which is homogeneous under G. We shall say that Ω is type B if it satisfies the following conditions:

(1) The G action is rationally homogeneous.

(2) As a submanifold of $\mathbf{C}^n$, Ω has a real analytic, codimensional-one, Levi nondegenerate boundary.

(3) The G action extends analytically to $\partial\Omega$ and $\partial\Omega$ is homogeneous under this action.

(4) There is a point $\omega_0 \in \partial\Omega$ which is an attractive fixed point for some $g_0 \in G$. Thus $g_0^n z \to \omega_0$ for all $z \in \Omega$.

Then we have the following partial converse to Theorem 3.

Theorem 5. *Every type* B *domain is type* I. *More precisely,* Ω *has a realization as a type* B *domain if and only if it is biholomorphic to a dilated nil-ball.*

We should comment on the term "dilated" appearing in this theorem since it will play a significant role in many of our results. As mentioned above, Siegel N-P domains are determined by a nilpotent Lie group N and a subgroup $P \subset N_c$. In the case of a dilated nil-ball, one has much additional structure, including a "dilation." This is a one-parameter subgroup $\delta(t)$ of semisimple Lie algebra automorphisms of the Lie algebra

$\mathcal{N}$. Dilations, by definition, have the property that each eigenvalue $\lambda_i(t)$ of $\delta(t)$ is of the form $t^{n(i)}$ for some *positive* integer $n(i)$. By replacing t by t^a, one can always choose the $n(i)$ to be a relatively prime set. In this case, the largest $n(i)$ is referred to as the dilation degree of the dilation. In the case of dilated nil-balls, we also refer to it as the "dilation degree of the domain" although it really is a function of the presentation of the domain as a homogeneous space.

One of the first conclusions we draw from Theorem 3 is the following:

Corollary 6. *Let Ω be a type* I *domain. Then the Koszul structure of Ω is Kähler if and only if Ω is pseudo-convex. In this case Ω is biholomorphic to the unit ball in $\mathbb{C}^n$.*

Corollary 6 allows us to assume that Ω is not pseudo-convex. We refer to the class of such type I domains as type I_n. Similarly, we define type B_n. From the homogeneity of the boundary, it follows that in the nil-ball realization, no point of the boundary of a domain of type I_n is pseudo-convex. Hence, every holomorphic function on an I_n domain extends past the boundary. Since the complement of $\overline{\Omega}$ is homogeneous under G_0, it follows that the hull of holomorphy is in fact $\mathbf{C}^n$. Thus we obtain the following corollary:

Corollary 7. *In the nil-ball realization, every holomorphic function on a type I_n domain extends to all of $\mathbf{C}^n$.*

We shall let $I(\Omega)$ denote the group of Koszul isometries of Ω, and G_Ω the identity component of the group of holomorphic isometries. It is known that both $I(\Omega)$ and G_ω are Lie groups which act analytically on Ω (see [8, p. 255, Theorem 32]). In [14], Webster proved that if a domain has a smooth, Levi nondegenerate, algebraic boundary, then every automorphism is algebraic. A globally defined, single valued algebraic mapping must be polynomial. Thus, we obtain:

Corollary 8. *If Ω is type I_n, then, in the nil-ball realization, G_ω acts polynomially.*

One of the more important consequences of Corollary 4 is that due to the nondegeneracy of the Levi form, we have the theory of the Chern-Moser normal form at our disposal [2]. We work in the nil-ball realization and choose a base point x_0 in the boundary (see §4 below). In [2], Chern and Moser showed that on a neighborhood of x_0, there is a biholomorphic mapping transforming $\partial\Omega$ into a set of the form in $\tau = r(z, \tau)$, where (z, τ) represents the general point of $\mathbf{C}^{n-1} \times \mathbf{C}$. The function r is real valued and real analytic and x_0 is transformed into $(0, 0)$. Furthermore, the power series expansion of r about $(0, 0)$ is of a very special form. For example, the first nonzero term is in degree 2 and is, in fact, the Levi form. The normal form is unique up to transformations of Ω by a certain group of linear fractional transformations.

One of the more striking properties of the type I_n domains is the following theorem which may be considered as a generalization of the Cartan result that a biholomorphism of a bounded domain which has a fixed point is linear. One should also compare this with the results of Ezhov [4], Webster [14], and Kruzhilin and Laboda [9].

Theorem 9. *Let Ω_1 and Ω_2 be type I_n domains in the nil-ball realization. Let A be a biholomorphic from Ω_1 to Ω_2 which preserves the origin. Let A' be the corresponding local biholomorphism of the Chern-Moser normal forms. Then A' is linear.*

In principle, this solves the equivalence problem for type I domains. To determine whether two such domains are locally equivalent, it is only necessary to check whether the normal forms are linearly equivalent. Of course, the difficulty lies in computing the normal forms. Surprisingly, for type B domains, the computation of the normal form is actually rather simple.

Given the Siegel pair of a type B_n domain, we describe an explicit normal form for the domain, which we call the *canonical form*. In the canonical form, the function r turns out to be a polynomial, independent of τ, of degree at most equal to the dilation degree of the domain. The transformation which transforms the domain from its realization as a nil-ball into canonical form is a polynomial operator with polynomial inverse. Both this operator and r are explicitly computable by means of an explicit algorithm. The canonical form is *global* in the sense that the mapping of $\partial\Omega$ onto this normal form establishes a biholomorphism of $\overline{\Omega}$ onto a domain $\operatorname{im}\tau \geq r(z)$.

One interesting consequence of these results is the following corollary, which follows from the fact that the normal form has no degree three terms. We remind the reader that a hyperquadratic domain is a domain in $\mathbf{C}^{n-1} \times \mathbf{C}$ defined by an equation of the form $\operatorname{im}\tau > H(z, z)$, where H is a nondegenerate Hermitian form on $\mathbf{C}^{n-1}$. Since hyperquadratics have dilation degree 2, this result says that dilation degree 3 is uninteresting.

Corollary 10. *Every type B_n domain with dilation degree $d = 3$ is biholomorphic to a hyperquadratic domain.*

Dilation degree 4 is also explicitly describable. Let Q be the fourth degree form on $\mathbf{C}^n$ defined by

$$Q(z) = \sum c_{\alpha,\beta} z^\alpha \overline{z}^\beta \qquad (|\alpha| + |\beta| = 4, \ |\alpha|\,|\beta| \neq 0), \tag{2}$$

where α and β, of course, represent multi-indices of length n. We shall assume that $c_{\alpha,\beta}$ is Hermitian symmetric in α, β so that Q is real valued. Let $0 \le s \le k$ be two integers. Let H_s be the Hermitian form on $\mathbf{C}^k$ defined by

$$H_s(z, w) = \sum_1^s z_i\overline{w}_i - \sum_{s+1}^k z_i\overline{w}_i .$$

Let $\Omega \subset \mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k \times \mathbf{C}^n$ be defined by the inequality

$$\text{im}\,\tau > \text{re}(z, w) + H_s(q, q) + Q(z), \tag{3}$$

where (τ, z, q, w) represents the general point of $\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k \times \mathbf{C}^n$, and $(\cdot, \cdot)$ is the usual Hermitian scalar product on $\mathbf{C}^n$. Then we have the following:

Corollary 11. *Every domain of the form given by* (3) *is homogeneous, of type* B, *with dilation degree* 4. *Conversely, every type* B *degree-*4 *domain has a realization in this form.*

Type B_n domains (by definition) have the property that the group acts transitively on the boundary. Thus, any biholomorphism is a product of a group element with an automorphism preserving the base points. When combined with Theorem 9, this has the following corollary. This corollary amounts to a complete solution of the problem of determining when two type B domains are equivalent.

Corollary 12. *Let the canonical forms of* $\partial\Omega_i$ *be defined in* $\mathbf{C}^{n-1} \times \mathbf{C}$ *by* $\text{im}\,\tau = r_i(z)$ *for* $i = 1, 2$. *Then* Ω_1 *and* Ω_2 *are biholomorphic if and only if there is a complex linear endomorphism of* $\mathbf{C}^{n-1} \times \mathbf{C}$ *which maps the function* $\text{im}\,\tau - r_1(z)$ *onto* $\text{im}\,\tau - r_2(z)$.

Of course, these results also apply when $\Omega_1 = \Omega_2$. In this case, they amount to an explicit computation of the automorphism group of the domain. Using Corollary 12 and Theorem 15 below, we give an example of a type B_n domain in $\mathbf{C}^4$ for which the full automorphism group is solvable. This is interesting because in [6], Koszul had conjectured that a Koszul domain was always symmetric. This conjecture was shown to be false by Pjatecki-Sapiro (see [12]—not all Siegel domains are symmetric). However, our example is particularly simple and dramatic.

Having considered the equivalence problem, the next natural collection of questions concern how abundant type I domains are. Ideally, what would be desired is a way of producing all possible polynomials r such that the domain $\tau > r(z)$ is the normal form of a type I domain. One would then ask to be able to produce from r a group that acts transtitively on the domain.

For general type I domains, we are still very far from this ideal. However, for a specialized class of domains, the "holomorphically abelian" domains, we have made considerable progress on these questions. This subclass is defined by the conditions that N acts simply transitively on $\partial\Omega$ and that the space of tangential Cauchy-Riemann operators is spanned over $C^\infty(\partial\Omega)$ be a commuting family of N-invariant differential operators. In terms of the Siegel pair N-P, this is equivalent to saying that P is abelian and that $P \cap N = \{e\}$. This class, while admittedly rather special, turns out to still be broad enough to contain vast classes of domains (see Theorem 15 below.)

It turns out that it is possible to construct the general holomorphically abelian domain in terms of certain abelian, associative, nilpotent algebras over $\mathbf{C}$. To describe this construction, let $\mathscr{A}$ be such an algebra. Assume also that we are given a Hermitian symmetric, nondegenerate, bilinear form H on $\mathscr{A}$. We define a (usually nonassociative) binary operation "$\circ$" on $\mathscr{A}$ by the equation

$$H(XY, Z) = H(X, Z \circ Y). \tag{4}$$

We shall say that $\mathscr{A}$ is a *duality* algebra if

$$H(X \circ Y, Z \circ W) = H(X \circ Z, Y \circ W) \tag{5}$$

for all X, Y, Z, and W in $\mathscr{A}$. This is equivalent with

$$Z \circ (X \circ Y) = Y \circ (X \circ Z). \tag{6}$$

Finally, we say that a one-parameter group of complex linear, semisimple automorphisms $\delta(t)$ of $\mathscr{A}$ is a homogeneity if $\delta(t)$ has only real eigenvalues and H is homogeneous of degree one under $\delta(t)$. We say that $\mathscr{A}$ is homogeneous if it has a homogeneity. If the eigenvalues of $\delta(t)$ are all of the form t^c, where $c > 0$, then the homogeneity is said to be of dilation type. In this case we say that $\mathscr{A}$ is dilated. It is easily seen that in this case $\mathscr{A}$ does indeed have a dilation (a one-parameter group $\delta(t)$ as above where all of the eigenvalues are t^n for some positive integer n).

Given a duality algebra $\mathscr{A}$, we define $[X, Y] = X \circ Y - Y \circ X$. It turns out that despite the nonassociativity of "$\circ$", this bracket satisfies the Jacobi identity. Thus, $\mathscr{A}$ becomes a real Lie algebra. Furthermore, we define $\phi = \operatorname{im} H$. Then ϕ is a Lie algebra cocycle of $\mathscr{A}$. We let $\mathscr{N} = \mathscr{A} \times \mathbf{R}$. We define a Lie structure on $\mathscr{N}$ by the stipulation

$$[(X, s), (Y, t)] = ([X, Y], \phi(X, Y)).$$

Let $\mathscr{P} \subset \mathscr{A}_c$ be the $-i$ eigenspace of J, where J is the complex multiplication in $\mathscr{A}$. We identify $\mathscr{P}$ with the subalgebra $\mathscr{P} \times \{0\}$ in $\mathscr{N}$.

Let N, N_c, and $P \subset N_c$ be the connected, simply connected Lie groups corresponding to $\mathcal{N}$, $\mathcal{N}_c$, and $\mathcal{P}$ respectively. Then, the main result of §5 is the following:

Theorem 13. *Given a homogeneous duality algebra* $(\mathcal{A}, H, \delta)$, *the corresponding Siegel pair* N-P *is the Siegel pair of a holomorphically abelian domain. Conversely, every holomorphically abelian domain* Ω *is associated to a Siegel pair* N-P *defined from a homogeneous, duality algebra as above. The correspondence between isomorphism classes of Siegel pairs and isomorphism classes duality algebras is bijective.*

The Chern-Moser normal form of the domain Ω corresponding to a dilated $\mathcal{A}$ is somewhat simpler than in the general type B_n domain case. We prove the following:

Theorem 14. *Assume that* $\mathcal{A}$ *is dilated and let* Ω *be the corresponding domain. A Chern-Moser canonical form of* $\partial\Omega$ *is describable as the set of points* $(q, w) \in \mathcal{A} \times \mathbf{C}$ *such that* $\operatorname{im} w = r(q)$, *where*

$$r(q) = H(q, q) - H(q \circ q, q \circ q) = 3H(qq, qq) + \cdots,$$

the dots representing a finite sum of terms of degree five or greater in q.

The dilation degree of $\mathcal{A}$ is the number of distinct eigenvalues of δ. If the dilation degree of the domain is 4 or less, then there are no terms with degree greater than 4 so this formula is exact. Comparing this form with (3), it is easily seen that holomorphically abelian domains have the property that Q is degree 2 in both z on $\overline{z}$. Thus, $c_{\alpha,\beta} = 0$ if either α or β is not of length 2.

This description has a partial converse. If Q is as described, we may write

$$Q(z) = \sum d_{\alpha,\beta} z_{a_1} z_{a_2} \overline{z}_{b_1} \overline{z}_{b_2}, \tag{7}$$

where $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$ range over the set of multi-indices with a_i and b_i between 1 and n. We shall assume that the coefficient $d_{\alpha,\beta}$ is symmetric in a_1, a_2 and in b_1, b_2 and is Hermitian symmetric in α, β. We shall say that the Hermitian signature of Q is the pair (u, v), where u and v are, respectively, the number of positive and the number of negative eigenvalues of the $n^2 \times n^2$ matrix $[c_{\alpha,\beta}]$. We say that the anti-Hermitian signature of Q is the corresponding pair for the matrix $[d_{\alpha,\beta}]$, where

$$d_{(a_1,a_2),(b_1,b_2)} = c_{(a_1,b_1),(a_2,b_2)}.$$

We say that s dominates the pair (u, v) if $u \le s$ and $v \le n - s$. Then we have the following:

Theorem 15. *Let Q be as in* (7) *above. Then the domain given by* (3) *is holomorphically abelian if s dominates either the Hermitian or the anti-Hermitian signature of Q.*

The organization of this work is as follows. In §1, we define the term rank and establish conditions for rationality. In §2, we prove Theorem 3. In §3, we study biholomorphisms between homogeneous nil-balls. §4 is devoted to the computation of the Chern-Moser normal forms, while in §5, we study the holomorphically-abelian domains.

1. Rank and rationality

In this section, unless otherwise stated, we shall always assume that X is type E and that the G action is effective. We shall also adopt the convention that Lie groups will be denoted by upper case italic letters and the corresponding Lie algebra will be denoted by the corresponding upper case script letter.

Our first goal is to prove Theorem 1. We begin by recalling some of the theory developed in [6]. Let X be a manifold which is homogeneous under G and which carries a G-invariant volume form. Let b_0 be a fixed point in X and let L be the isotropy subgroup of b_0. Koszul constructed a functional $\lambda \in \mathscr{G}^*$ and a complex subalgebra $\mathscr{Q} \in \mathscr{G}_c$ such that

(1) $\mathscr{Q} + \overline{\mathscr{Q}} = \mathscr{G}_c$,
(2) for all $W \in \mathscr{Q}$, $[W, \mathscr{Q}] \subset \ker \lambda$,
(3) $\mathscr{Q} \cap \mathscr{G} = \mathscr{L}$,
(4) $\mathscr{L} \subset \ker \lambda$,
(5) $\mathrm{ad}^*(L)\lambda = \lambda$.

We shall refer to such triples $(\mathscr{G}, \mathscr{Q}, \lambda)$ as Koszul data for G/L, even when λ is not the explicit functional constructed by Koszul. The Koszul data is *nondegenerate* if the converse to (2) also holds. This condition is equivalent to the nondegeneracy of the form $B_\lambda(X, Y) = \lambda([X, Y])$ on $\mathscr{G}/\mathscr{L}$ (the Koszul form). Note that in this case, the center of $\mathscr{G}$ is in $\mathscr{L}$. The effectiveness of the action then implies that the center is trivial. Thus any subgroup of $\mathscr{G}$ which acts transitively on X will have a discrete center.

The geometric interpretation of the Koszul data is that $\mathscr{G}_c/\mathscr{L}_c$ is the complex tangent space of $X = G/L$ at b_0. The space $\mathscr{Q}/\mathscr{L}_c$ definites the space of vectors of type $(0, 1)$. The space $\mathscr{L}_c$ is precisely the radical of the form B_λ. This form projects to a form on $\mathscr{G}/\mathscr{L}$ which is the pseudo-Kählerian form if X is a Koszul manifold.

We shall require a formula for a specific λ. Up to a normalization factor, λ will be the functional constructed in formula (4.5) of [6]. We refer to this λ as the Koszul function. We begin by observing that there is a linear operator $J: \mathscr{G} \to \mathscr{G}$ such that for all $Y \in \mathscr{G}$, $Y - iJY \in \mathscr{Q}$. This operator is uniquely defined Modulo $\mathscr{L}_c$. Assume that such an operator has been chosen. For $Y \in \mathscr{G}$, we define $T_Y = \mathrm{Ad}\, JY - J\,\mathrm{Ad}\, Y$. As Koszul remarks, T_Y leaves $\mathscr{L}$ invariant. Formula (4.5) of [6] implies that we may define λ by

$$\lambda(Y) = -(n+1)^{-1}\,\mathrm{Tr}_{\mathscr{G}/\mathscr{L}}\, T_Y. \tag{8}$$

The functional λ is independent of the choice of J [6, p. 570]).

Given a triple $(\mathscr{G}, \mathscr{Q}, \lambda)$ as above, it is possible to define a complex, pseudo-Kählerian manifold which will be of importance to us. Let G_c be a connected complex Lie group with Lie algebra $\mathscr{G}_c$. Let Q_0 be the connected subgroup of G_c corresponding to $\mathscr{Q}$. We define Q to be the set of $x \in G_c$ which normalize Q_0 and satisfy $\mathrm{ad}^* x(\lambda)|\mathscr{Q} = \lambda|\mathscr{Q}$. It is easily seem that Q_0 is the component of the identity of Q. We let $L_1 = Q \cap G$. Then L_1 is the set of elements in the isotropy subgroup of λ which normalize $\mathscr{Q}$.

Now, assume that λ is the Koszul functional. Then just normalizing $\mathscr{Q}$ is sufficient to belong to L_1 (see §2.2 of [5]). Thus, in this case, our L_1 is the same as Hano's K_1. We define $X_1 = G/L_1 = GQ/Q \subset G_c/Q$. This manifold is the open subset of a projective variety considered by Hano. Hano proves (following Koszul) that both X_1 and X are G-equivariant covering spaces of the coadjoint orbit in $\mathscr{G}^*$ defines by λ. Now, since X is type E, we may take G to be exponential solvable. It is known (see [1]) that all coadjoint orbits of an exponential solvable Lie group are simply connected and diffeomorphic with $\mathbf{R}^n$. Thus, $X = X_1$ is diffeomorphic with $\mathbf{R}^n$. In particular, X is contractible. It follows that even if G is not exponential, L is connected and $L = L_1$.

It now follows from Hano's theorem that X is biholomorphic with an open subset of a closed complex variety in $\mathbf{P}^r$ for some r and that G_ω is the identity component of a real algebraic group which acts birationally on X. In fact, this algebraic group is the Zariski closure of the image of G_ω under the adjoint representation. In particular, the adjoint representation is an isomorphism, proving that G_ω is centerless.

Since X is also contractible, the arguments of [11] (from Lemma (2.4) on p. 404 to the top paragraph on p. 406) apply to X, proving that X is realizable as a Siegel domain of type N-P. Explicitly, let G_c be the complex algebraic closure of G_ω. The arguments from [11] prove that there is a completely solvable real algebraic subgroup $S \subset G_\omega$ which acts transitively on X. Furthermore, if N_c is the complex algebraic closure of the nilradical of S, then $S_c \subset N_c Q$. Transitivity proves that $G \subset SQ$. Hano showed that GQ is open in G_c. It follows that $N_c Q$ is also open in G_c. Consider the following sequence of containments of open sets:

$$X = S/S \cap Q = SQ/Q \subset N_c Q/Q \subset G_c/Q.$$

The last containment is Hano's embedding of X as an open subset of a projective variety. The second to last containment is (essentially) the realization of X as a Siegel N-P domain since $N_c Q/Q = N_c/N_c \cap Q = \mathbf{C}^n$ (from nilpotence). The fact that G_ω acts rationally in the Siegel realization follows from the fact that the last containment is also an algebraic embedding. This finishes the proof of Theorem 1. q.e.d.

Let ρ denote the adjoint representation of G_ω on $\mathscr{G}_\omega$. From Theorem 1, G_ω is isomorphic with $\rho(G_\omega)$, and $\rho(G_\omega)$ is a "semialgebraic" group (the Euclidean identity component of a real algebraic group.) Before proving Theorem 2, we prove the following:

Lemma 16. *Let G be a subgroup of G_ω which acts transitively on X. Suppose that G is* ad-*algebraic. Then $\rho(G)$ is a semialgebraic subgroup of $\rho(G_\omega)$.*

Proof. Let G^a be the identity component of the algebraic closure of $\rho(G)$. Then G^a leaves $\mathscr{G} \subset \mathscr{G}_\omega$ invariant. Furthermore, from the ad-algebraic condition, $G^a|\mathscr{G} = \rho(G)|\mathscr{G}$. It follows that $G^a = Z\rho(G)$, where Z is the kernel in G^a of restriction to $\mathscr{G}$. Note that Z centralizes $\rho(G)$ since it centralizes $\mathscr{G}$. But then Z must centralize G^a. It follows from the Koszul condition that Z is discrete and, hence, finite. It follows that $\rho(G)$ is the component of the identity in G^a, proving our lemma. q.e.d.

Armed with this result, we may now prove Theorem 2. According to the above lemma, we may consider G as an algebraic group. We may write $G = M \times_s G_u$, where M is reductive and G_u is the unipotent radical. We may then write $M = ANK$, where K is maximal compact in G, A is an $\mathbf{R}$-split torus, and N is nilpotent. Note that $H_*(G) = H_*(K)$, where H_* is the homology complex. On the other hand, $G/L = X$ is just $\mathbf{R}^n$. Thus $H_*(G) = H_*(L)$. It follows that the dimension of the maximal compact subgroup of L is the same as that of K. Since all maximal compact subgroups of G are conjugate, it follows that ANG_u acts transitively on X. This proves Theorem 2. q.e.d.

Next we define rank. Again, we assume that G is ad-algebraic and that X is type E. Then there is a maximal triangular subgroup S of G which acts transitively. We have the equality $G = SL$. Every maximal triangular subgroup is an L-conjugate of S. Thus, every maximal triangular subgroup acts transitively. We define the rank of the G action on G/L to be the dimension of S/S_uL_S, where $L_S = L \cap S$ and S_u is the unipotent radical of S. This is clearly independent of the choice of S. It is also easily seen to be independent of the choice of base point in G/L. Due to the next proposition, this is also independent of the choice of G. Thus, we may call this number the "rank" of X.

Proposition 17. *Suppose that G is* ad*-algebraic and that X is type* E. *Then the rank of the G_ω action on X is the same as that of the G action.*

Proof. Lemma 16 tells us that G is a semialgebraic subgroup of G_ω. By definition, the rank of the G action is the same as the rank of a maximal triangular subgroup of G. Thus, we may assume that G is triangular. More precisely, the proof of Lemma 16 shows that we may take G to be the semidirect product of an algebraic torus with a unipotent group. Hence we may assume that G is triangular *in* G_ω. It follows that $G \subset S$, where S is a maximal triangular subgroup of G_ω. We may also choose a maximal $\mathbf{R}$-split torus T_G of G contained in a maximal $\mathbf{R}$-split torus T_S of S. Let L be the isotropy subgroup of b_0 in S and let T_L be a maximal $\mathbf{R}$-split torus in L. From the conjugacy of tori, there is a conjugate of T_L contained in T_S. Conjugating L is equivalent with changing the base point in X which does not change the rank. Thus, we may assume that $T_L \subset T_S$. It is easily seen that the G_ω rank of X is $\dim T_S/T_L$ and the G rank is $\dim T_G/(T_G \cap T_L)$. Since $GL = S$, it also follows that $T_S = T_G T_L$. This proves the proposition. q.e.d.

Rank-0 manifolds are easily described. By definition a rank zero manifold is a homogeneous space of a nilpotent Lie group N.

Proposition 18. *Let W be a complex, simply connected manifold which is homogeneous under a real nilpotent Lie group N. Assume that W has an invariant volume form. Then the Koszul functional λ is identically zero.*

Proof. Let $b_0 \in W$ be a fixed base point and let $(\mathscr{N}, \mathscr{Q}, \lambda)$ be the corresponding Koszul data. Let Z_0 be a nonzero, central element of $\mathscr{N}$. Clearly, $\lambda(Z_0) = 0$. If $Z_0 \in Q$, then we form the quotient of $\mathscr{N}$ by the ideal generated by Z_0. The corresponding group still acts transitively on W and the Koszul functional is the projection of λ to the quotient. By induction, we may assume that this projection is 0, proving the lemma in this case.

Thus, we may assume that Z_0 is not in $\mathscr{Q}$. Let $Y_0 = JZ_0$. Note that $W = Y_0 + iZ_0$ belongs to $\mathscr{Q}$. It follows that ad Y_0 normalizes $\mathscr{Q}$. This is equivalent to saying that for all Y, $T_Y(Y_0)$ is zero $\mod \mathscr{L}$.

Now, let $\mathcal{Q}' = \mathcal{Q} + \mathbf{C}Z_0$. let $\mathcal{M}' = \mathcal{Q}' \cap \mathcal{N}$. This space is spanned by $\mathcal{L}$, Z_0, and Y_0. The group N acts transitively on $W' = N/M'$. This space is a complex manifold with complex structure defined by $\mathcal{Q}'$. By induction, its Koszul functional may be assumed trivial. However, from (8) and the previous paragraph, this functional is a nonzero multiple of λ, proving the proposition.

2. Theorem 3

We shall continue the conventions established in the first paragraph of §1.

Let us now recall the definition of nil-ball. Let $\mathcal{N}$ be a nilpotent Lie group with Lie algebra $\mathcal{N}$. Let $\lambda \in \mathcal{N}^*$. A complex subalgebra $\mathcal{P}' \subset \mathcal{N}_c$ is a totally complex polarization for λ if it satisfies the following properties:

(a) $Z \in \mathcal{P}'$ if and only if $[Z, \mathcal{P}'] \subset \ker\lambda$.

(b) $\mathcal{P}' + \overline{\mathcal{P}'} = \mathcal{N}_c$.

Given a totally complex polarization, there is a canonical way of associating a domain with it. Explicitly, we let $\mathcal{P} = \mathcal{P}' \cap \ker\lambda$. By forming a quotient, we may assume that the kernel of λ contains no nontrivial ideals. We refer to this condition as "effectivity." In this case, the center $\mathcal{Z}$ of $\mathcal{N}$ is one dimensional and λ is nontrivial on $\mathcal{Z}$. Let Z_0 be a basis for the center, normalized by the condition that $\lambda(Z_0) = 1$. Then, clearly,

$$\mathcal{N}_c + \mathcal{N} + i\mathbf{R}Z_0 + \mathcal{P}.$$

Let $X = N_c/P$. From the nilpotency of N_c, this space is biholomorphic to $\mathbf{C}^n$ for some n. The above equality implies that the image of N in X under the quotient map is a real codimension-one submanifold which divides X into two connected components

$$\Omega^{\pm} = N(\exp i\mathbf{R}^{\pm}Z_0)P/P$$

(see Lemma 29 below).

We refer to $\Omega^+ = \Omega$ as the domain associated to the polarization. (Note that Ω^- is the domain associated to $(\mathcal{P}, -\lambda)$.) Such domains are what we refer to as the nil-balls. The boundary of each of these domains is smooth and equals N/K, where $\mathcal{K} = \mathcal{P} \cap \mathcal{K}$. Note also that, in general, the N orbits in Ω are of real codimension-one so that such domains are not necessarily homogeneous. However, suppose that $t \to \delta(t)$ is a one-parameter group of semisimple, $\mathbf{R}$-split automorphisms of $\mathcal{N}$. We shall say that the pair $(\lambda, \mathscr{P}')$ is homogeneous if $\delta(t)$ preserves $\mathscr{P}'$ and $\delta(t)^*(\lambda) = t\lambda$ for all t. It is easily seen that under these assumptions, each of the nil-basis is homogeneous under the group $G = \mathbf{R} \times_s N$, where $\mathbf{R}$ acts on N by means of δ. If all of the eigenvalues of $\delta(t)$ are of the form t^c for $c > 0$, then we say that the nil-ball is dilated.

We should remark that the definition of "dilated" has a different (more common) formulation. Let $t \to \delta(t)$ be a one-parameter group of semisimple, $\mathbf{R}$-split automorphisms of $\mathscr{N}$. Then δ is a dilation if its eigenvalues are all of the form $t^{n(i)}$, where the $n(i)$ are positive *integers*. The dilation is said to preserve $(\lambda, \mathscr{P}')$ if $\mathscr{P}'$ is invariant under δ and $\delta(t)^*(\lambda) = t^d\lambda$ for some integer d. (Note that in this case, t^d must be the largest eigenvalue of $\delta(t)$.) It is a theorem that the nil-ball is dilated in the sense defined above if and only if there exists a dilation preserving the pair $(\lambda, \mathscr{P}')$. The equivalence follows easily from the observation that the set of all automorphisms of $\mathscr{N}$ which are scalar on the eigenspaces of the homogeneity is an algebraic torus and each eigenspace defines an algebraic character of this group. We shall leave the details of the proof as an exercise for the reader. (We are indebted to Roger Howe for pointing this equivalence out to us.) For a given dilation δ, the integers d is called the dilation degree. (It depends upon the choice of dilation.)

In order to prove Theorem 3, we must first prove that homogeneous nil-balls are rank-one Koszul domains. We note that Z_0 is an eigenvector of $\delta(t)$ with eigenvalue t. We shall describe what will turn out to be the Koszul data for Ω relative to the base point $b_0 = (\exp iZ_0)P$. Let $\mathscr{G}$ be the Lie algebra of G and let $A \in \mathscr{G}$ be the infinitesimal generator of δ. Then $[A, Z_0] = Z_0$. Let β be the functional on $\mathscr{G}$ which equals λ on $\mathscr{N}$ and is zero at A. Let $\mathscr{Q}$ be the subalgebra spanned by $\mathscr{P}$ and $A - iZ_0$.

The following proposition is one half of Theorem 3.

Proposition 19. *The domain Ω^+ is a rank-one Koszul domain with Koszul data $(\mathscr{G}, \mathscr{Q}, \beta)$. The functional β is the Koszul functional of* (8) *above.*

Proof. Clearly, the G action is rank one. It is also easily seen that $\mathscr{Q}$ is the subalgebra referred to in the Koszul data. Furthermore, the form B_β defined from β is clearly nondegenerate. The only issue is to prove that β is the Koszul functional λ'. We shall make use of (8).

From the definition of $\mathscr{Q}$, we may assume that $JA = Z_0$. Furthermore, there is an Ad A invariant direct sum decomposition

$$\mathscr{P} = \mathscr{P}^+ \oplus \mathscr{L}_c.$$

We may choose J so that $\mathscr{P}^+$ is in the $+i$ eigenspace of J, and $\mathscr{L}_c$ is in the kernel of J.

Lemma 20. $\mathrm{Tr}_{\mathscr{G}/\mathscr{L}} \mathrm{Ad}\, A = n$.

Proof. Note first that B_λ defines a nondegenerate bilinear form on $\mathscr{N}/(\mathscr{L}+\mathscr{Z})$. Furthermore, $\delta(t)^* B_\lambda = tB_\lambda$. It follows that $(\mathrm{Ad}\, A)^* B_\lambda = B_\lambda$. Hence,

$$\mathrm{Tr}_{\mathscr{N}/(\mathscr{L}+\mathscr{Z})} \mathrm{Ad}\, A^* = \tfrac{1}{2}\dim(\mathscr{N}/(\mathscr{L}+\mathscr{Z})).$$

Our lemma follows from this and the observations that Z_0 is an eigenvector of $\mathrm{Ad}\, A$ of eigenvalue 1 and $n = \dim \Omega^+ = (\dim \mathscr{N}/\mathscr{L} + 1)/2$.

It follows that $\lambda'(Z_0) = 1$. It is also easily seen that $\lambda'(A) = (n+1)^{-1} \mathrm{Tr}\, J\, \mathrm{Ad}\, A$. We claim that this is zero. To see this, we write

$$\mathscr{G}/\mathscr{L} = \mathscr{G}_0 \oplus (\mathscr{P}^+ + \overline{\mathscr{P}}^+) \cap \mathscr{G},$$

where $\mathscr{G}_0$ is the span of A and Z_0. This decomposition is invariant under both J and $\mathrm{Ad}\, A$. The trace of $J\,\mathrm{Ad}\, A$ on $\mathscr{G}_0$ is 0. On the second direct summand, $\mathrm{Ad}\, A$ and J commute. This, $J\,\mathrm{Ad}\, A$ has only purely imaginary eigenvalues on this space, proving our claim. q.e.d.

To finish the proof of the proposition, we need only prove the following:

$$(9) \qquad (\mathscr{P} + \overline{\mathscr{P}}) \cap \mathscr{G} \subset \ker \lambda'.$$

Suppose that $X \in \mathscr{P} + \overline{\mathscr{P}}$. Then both X and JX belongs to $\mathscr{N}$. It follows that $T_X(A)$ belongs to $\mathscr{N}$ and $T_X(Z_0) = 0$. Thus, for such X,

$$\mathrm{Tr}_{\mathscr{G}/\mathscr{L}} T_X = \mathrm{Tr}_{\mathscr{N}/\mathscr{M}} T_X,$$

where $\mathscr{M} = \mathscr{L} + \mathbf{R}Z_0$.

Now, let $P' = P(\exp \mathbf{C}Z_0)$. Then N acts homogeneously on $X' = N_c/P'$. The functional defined by the right side of the above is the Koszul functional for X' which is zero from Proposition 18. q.e.d.

Now we turn to the converse statement in Theorem 3. Suppose that X is a type I manifold with Koszul data $(\mathscr{G}, \mathscr{Q}, \lambda)$. From Theorem 1, we may assume that G is algebraic, completely solvable with codimension one nil-radical. We will also assume, as usual, that G acts effectively. The subalgebra $\mathscr{Q}$ is also algebraic. We shall let $T = \exp \mathbf{R}A$ be a maximal torus of G so that $G = TN$, where N is the nilradical of G.

Let Q be the connected subgroup of G_c corresponding to $\mathscr{Q}$. Since coadjoint orbits of G are simply connected, $Q \cap G$ is the isotropy subgroup of X. Thus we may identify X with GQ/Q in G_c/Q. This, however, is not the most convenient realization of X. Since $\mathscr{Q} + \mathscr{N}_c = \mathscr{G}_c$, the maximal torus of $\mathscr{Q}$ is also maximal in $\mathscr{G}_c$. Thus there is an $x_0 \in N_c$ such that $\widetilde{Q} = x_0^{-1} Q x_0$ contains T. Let $P = \widetilde{Q} \cap N_c$ so that $\widetilde{Q} = T_c P$.

We may realize X in $Y = G_c/\tilde{Q} = N_c/P$. (Note that Y is biholomorphic to $\mathbf{C}^n$.) We shall denote the image of X in Y by Ω. It is important to note that if we take $xP \in \Omega$ as the base point, then the subalgebra corresponding to $\mathscr{Q}$ becomes $\operatorname{ad} x(\tilde{\mathscr{Q}})$. We denote this algebra by $\tilde{\mathscr{Q}}_x$.

Let $\mathscr{Z} \in \mathscr{N}$ generate a complex, one-dimensional ideal $\mathscr{I}$ in $\mathscr{G}_\varepsilon$. Such elements exist because of the complete solvability of $\mathscr{G}$. For $x \in N_c$, let $\mathscr{P}_x = \operatorname{ad} x(\mathscr{P})$.

Lemma 21. *$iZ \notin \mathscr{N} + \mathscr{P}_x$ for any $x \in N_c$ such that $xP \in \Omega$.*

Proof. Assume that the lemma is false. Then $iZ \in \mathscr{N} + \mathscr{P}_x$ for some x such that $xP \in \Omega$. Conjugating by the general element of G and using homogeneity, it follows that the containment holds for all x such that xP belongs to Ω. Since the action is effective, $\mathscr{I}$ is not in $\mathscr{P}_x$ for any x. Let $Y' = N_c/IP$, where $I = \exp \mathscr{I}$, and let Ω' be the image of Ω in Y' under the quotient mapping. There is a holomorphic, direct sum decomposition $Y = Y' \times \mathbf{C}$ such that the I action on Y becomes translation in $\mathbf{C}$. (Note that I is central in N_c.) We claim that, in this decomposition, Ω is $\Omega' \times \mathbf{C}$. Once this is shown, it will follow that the function K of formula (1) is constant in the $\mathbf{C}$ variable, contradicting the Koszul condition.

To establish our claim, it suffices to show that Ω is invariant under the I action on Y. However, let $i \in I$ and $xP \in \Omega$. Using the centrality of i, we may write $i = np$ with n in N and p in P_x. Then $ixP = nxP \in \Omega$, proving the lemma.

Lemma 22. *We may choose Z so that $\Omega = N(\exp i\mathbf{R}^+ Z)P/P$.*

Proof. For $xP \in \Omega$, $\tilde{\mathscr{Q}}_x + \overline{\tilde{\mathscr{Q}}_x} = \mathscr{G}_c$. Hence, $\mathscr{P}_x + \overline{\mathscr{P}_x}$ is at most complex codimension two in $\mathscr{G}_c$ and codimension one in $\mathscr{N}_c$. It follows that $\mathscr{N} + \mathscr{P}_x$ is real codimension one in $\mathscr{N}_c$. Then iZ spans a complement over $\mathbf{R}$. Our lemma follows from Proposition (A.6) of [11]. q.e.d.

Let $x_0 = \exp iZ$. We normalize A so that $[A, Z] = Z$. Due to the above lemma, we may choose x_0P as the base point for Ω in Y. This has the effect that $\mathscr{Q} = \operatorname{ad} x_0^{-1}\tilde{\mathscr{Q}}$. This space is spanned by $\mathscr{P}$ and $W = \operatorname{ad} x_0(A) = A - iZ$. From the proof of (9) above, $\mathscr{P} \subset \ker\lambda$.

Next, we set $\mathscr{P}' = \mathscr{P} + \mathbf{C}Z$. Clearly, $\mathscr{P}'$ has the same dimension as $\mathscr{Q}$ and $[\mathscr{P}', \mathscr{P}'] \subset \ker\lambda$. It follows that $\mathscr{P}'$ is a complex polarization for λ. It is clear that Ω is the nil-ball defined from $(\mathscr{N}, \mathscr{P}', \lambda/\mathscr{N})$. This finishes the proof of Theorem 3. q.e.d.

Now we consider Corollary 6. We continue the notation established above. Let us first assume that Ω is Kähler in its Koszul structure. This is equivalent to saying that the following form H_λ is positive semidefinite on $\mathscr{Q}$:

$$H_\lambda(Z, W) = -\frac{i}{2}\lambda([Z, \overline{W}]).$$

Then H_λ is also positive on $\mathscr{P}$. It follows that $\mathscr{P}'$ is a positive polarization for $\beta = \lambda|\mathscr{N}$. The structure of totally complex positive polarizations for nilpotent Lie groups is well understood (see [1]). It is known that $\mathscr{P} \cap \overline{\mathscr{P}} = \mathscr{R}$ is an ideal in $\mathscr{N}_c$ which contains $[\mathscr{N}_c, \mathscr{N}_c]$. This space is actually an ideal in $\mathscr{G}$ since it is obviously $(\operatorname{Ad} A)$-invariant. Let $\mathscr{K} = \mathscr{R} \cap \ker\beta$. Then (since $\operatorname{Ad}^* A(\lambda) = \lambda$) $\mathscr{K}$ is also an ideal in $\mathscr{G}$. Thus, from the effectiveness of the action, $\mathscr{K} = 0$. We conclude that $\mathscr{N}$ is a two-step nilpotent Lie group with one-dimensional center and positive polarization. It is easily seen that the corresponding nil-ball is just the classical unbounded realization of the unit ball. This proves one part of our corollary.

The converse is similar. From the arguments on p. 406–407 of [11], the form H_λ on $\mathscr{P}$ is essentially the Levi form for the given nil-ball. Thus, if Ω is pseudo-convex, then H_λ is positive and $\mathscr{P}'$ is a positive polarization for λ. We then reason as before. This finishes the proof of Corollary 6. q.e.d.

Now we turn to the proof of Theorem 5. We remark that it is a consequence of the contractability of Ω that there is a solvable subgroup of G which acts transitively on Ω (see Proposition (2.6) of [11]). In the process of the proof we will need to change the group G several times. Each time, we shall need to verify that G has an appropriate element g_0.

Lemma 23. *We may assume that G is a real algebraic, completely solvable group and that ν is rational as a mapping of $G \times \Omega$ into $\mathbf{C}^n$.*

Proof. From Theorem (0.1) of [11], we may assume that G is a real algebraic group and that the map ν is rational in both variables. The point is to prove the complete solvability. Let g_0 be the element hypothesized in the definition of type B. Let $g_0 = an$ be the Jordan decomposition of g_0, where n is nilpotent, a is semisimple, and g_0 and a commute. We may write $a = a_1 a_2$, where $\operatorname{ad} a_2$ has pure imaginary eigenvalues on $\mathscr{G}_c$ and $\operatorname{ad} a_1$ has only real eigenvalues, and where a_1 and a_2 commute with each other and with n. The element a_2 generates a precompact subgroup. It follows that the element $a_1 n$ satisfies the same assumptions as g_0.

There is a maximal triangular subgroup S of G containing $a_1 n$. From the proof of Theorem (0.1) of [11], it follows that S acts transitively on Ω. This finishes the proof of the lemma. q.e.d.

For the remainder of this section, G will be assumed to be as stated above. Furthermore, we will assume that a maximal, $\mathbf{R}$-split torus A of

G has been chosen so that A contains the semisimple part a_o of g_0. Then, in the "AN" decomposition of G, $g_0 = a_0 n_0$, where a_0, n_0, and g_0 all commute. Let G_c be the complex algebraic group which has G as its real points. Let N_c be the unipotent radical of G_c. From Proposition (2.6) of [11], there is a complex subgroup Q of G_c which contains a maximal torus of G_c such that $K = Q \cap G$ is the isotropy subgroup of some point in Ω. The equality below yields a realization of Ω in N_c/R, where $R = Q \cap N_c$. Note that from nilpotence, $N_c/R \approx \mathbf{C}^n$:

$$\Omega = G/(Q \cap G) = GQ/Q \subset G_c/Q = N_c Q/Q = N_c/R.$$

Let $\phi\colon \Omega \to G_c/Q$ be the above described biholomorphism. Theorem (0.2) of [11] says that ϕ extends to a rational, G-equivalent, biholomorphism of a complex Zariski open subset of $\mathbf{C}^n$ onto a Zariski open subset of N_c/R. The domain of ϕ must contain boundary points of Ω. From the homogeneity of the boundary, it follows that ϕ is holomorphic on $\overline{\Omega}$. We shall also let ϕ denote the extension of ϕ to $\overline{\Omega}$.

Next we shall show that $\mathscr{N}$ carries a dilation. First, however, we shall need to change the realization of Ω in order to bring our notation into conformality with that of [11]. Since Q contains a maximal torus of G_c, some conjugate Q' of Q contains the real torus A. Let $P = Q' \cap N_c$. As described in [11, p. 405], we may realize Ω in $N_c/P = \mathbf{C}^n$. We let Ω' denote the corresponding subset of N_c/P. To describe the G action on Ω', let $x \in G$, $x = na$ with $n \in N$ and $a \in A$. Then, for all $mP \in \Omega'$,

$$na(mP) = n(m)^a P, \tag{10}$$

where $m^a = ama^{-1}$ (see [11, loc.cit]).

We shall need to evoke the structure theory developed in [11]. Let $\{X_1, \cdots, X_{2m-1}\}$ be a Jordan-Hölder basis $\mathscr{N}$ indexed by odd integers. We assume that the X_j are joint eigenvectors of A. For even m, let $X_{m+1} = iX_m$. Let $E\colon \mathbf{R}^{2m} \to N_c$ be the mapping

$$E(x_1, \cdots, x_{2m}) = (\exp x_1 X_1) \cdots (\exp x_{2m} X_{2m}). \tag{11}$$

Let $\mathscr{N}_j$ be the span of $X_j, \cdots, X_{2m}$ so that each $\mathscr{N}_j$ is a real ideal of $\mathscr{N}_c$. If $\mathscr{L}$ is any real linear subspace of $\mathscr{N}_c$, we set $\mathscr{L}_j = \mathscr{L} + \mathscr{N}_j$. The jump set $J(\mathscr{L})$ is the set of indices j such that $\mathscr{L}_j \neq \mathscr{L}_{j-1}$.

Now, suppose that $\mathscr{L} = \mathscr{P}$. Let $\mathscr{C}(\mathscr{P}) \subset \mathbf{R}^{2m}$ be the set of points x whose ith coordinate is zero for all $i \notin J(\mathscr{P})$. It follows from Proposition (A.2) of [11, p. 409] that $T = E(\mathscr{C}(\mathscr{P}))$ is a transversal to P in

N_c. Furthermore, $E^{-1} = (x_1, \cdots, x_{2m})$ defines global coordinates on N_c and the restriction of E^{-1} to T defines global coordinates for N_c/P. We denote these coordinates by $(y_1, \cdots, y_s)$, where $d = 2n$. (Recall that $\Omega \subset \mathbf{C}^n$.)

In these coordinates, N_c acts in a rather specific fashion. Let $JH(\mathscr{P}) = (j_1, j_2, \cdots, j_r)$ be the sequence of jump indices in increasing order. Let $n = (n_1, \cdots, n_{2m})$ and $y = (y_1, \cdots, y_d)$ be, respectively, coordinates of points in N_c and N_c/P. According to [9], for $r = 1, \cdots, d$, there are polynomials Q_r such that the N_c action on N_c/P is given by

$$(ny)_r = n_{j_r} + y_r + Q_r(n_1, \cdots, n_{j_r-1}, y_1, \cdots, y_{r-1}).$$

On the other hand, the A action is "diagonal" since each of the X_j are eigenvectors of A. Thus, for each $a \in A$, there are numbers $a_1, \cdots, a_d$ such that

$$y^a = (a_1 y_1, \cdots, a_d y_d).$$

Now let $g_0 = a_0 n_0$ be as described following the proof of Lemma 23. We also let w_0 be the corresponding attractive fixed point in $\partial\Omega$. For brevity of notation, we set $g = g_0$, $n = n_0$, and $a = a_0$.

Proposition 24. *The numbers a_i corresponding to a are all in the interval $(0, 1)$. Furthermore, $\phi(w_0) = P$, the identity coset in N_c/P.*

Proof. Let $y \in N_c/P$ represent an element of Ω', $y = (y_1, \cdots, y_d)$ Then, in terms of coordinates

$$\begin{aligned}(g^k y)_r &= (n^k a^k y)_r = (n(k) a^k y)_r \\ &= n(k)_{j_r} + a_r^k y_r + Q_r(n(k)_1, \cdots, n(k)_{j_r-1}, a_1^k y_1, \cdots, a_{r-1}^k y_{r-1}).\end{aligned}$$

Now the limit of $g^k y$ as $k \to \infty$ must exist and be independent of y for all $y \in \Omega'$. Since Ω' is open, for each r between 1 and d, there are x and y in Ω' with $x_j \neq y_j$ if and only if $j = r$. Subtracting $(g^k y)_r$ from $(g^k x)_r$, we see that $a_r^k \to 0$ as $k \to \infty$. It follows that each of the eigenvalues a_r is less than 1. Next, we use the commutativity of a and n to write $g^k y = a^k(n^k y)$. In coordinates, the expression $n^k y$ is growing at most polynomially in k while the a^k part forces an exponential decay. Clearly, the limit is zero, proving our proposition. q.e.d.

We note that from the proposition, for all y in N_c/P, $a_0^k y$ converges to the identity coset. We may therefore replace g_0 by a_0 and hence assume that g_0 is semisimple.

Next, we wish to show that the nilradical may be taken to be codimension one. For this, we shall use the smoothness of Ω at w_0 as well as some more structure theory. Note that from Proposition 24, Ω' is smooth at the identity coset.

Now, in [11, Lemma (A.4)], we showed that

$$e = J(\mathcal{N} + \mathrm{ad}\, x(\mathcal{P}))$$

is independent of x for $xP \in \Omega'$. Since the given Jordan-Hölder basis of $\mathcal{N}_c$ contains a Jordan-Hölder basis of $\mathcal{N}$, it is easily seen that $e \subset J(\mathcal{P})$. We reorder our basis so that the elements of e come last. We define a coordinate mapping $E'\colon \mathbf{R}^{2m} \to N_c$ by means of $(11)_n$ except now the product is taken with respect to the new order. A simple inductive proof shows that this still defines global coordinates for N_c. Furthermore, we obtain coordinates for N_c/P in precisely the same manner as before. For the sake of the next lemma, let us note that we may multiply any subset of the X_i by -1 without affecting our ability to us them to produce coordinates. We refer to this as redirecting the basis.

Lemma 25. *Let Ω_0 be the intersection of Ω' with the set T_0 of points y such that $y_i = 0$ for all $i \leq d-k$, where k is the length of e. Then, after possibly redirecting the basis defining E', Ω_0 may be described as the subset of N_c/P consisting of all points $y \in T_0$ such that $y_i > 0$ for all $d-k < i \leq d$.*

Proof. This all follows from Propositions (A.6) and (A.2) of [11].

Corollary 26. *In the above lemma, $k = 1$.*

Proof. Assume that $k > 1$. Let T_1 denote the set of points in N_c/P with $y_i = 0$ for all i not equal to d and $d-1$. Then the intersection Ω_1 with T_1 is the first quadrant in (y_{d-1}, y_d) space. By assumption, Ω has an analytic defining function r on a neighborhood of the identity coset in N_c/P. Thus, the set Ω_1 in T_1 is describable locally in the form $r > 0$ for some analytic function r on T_1. However, it is a simple power series argument that no analytic defining function can describe a quadrant at its vertex. q.e.d.

It now follows from Proposition (A.6) of [11] that Ω' is the image of $N \exp \mathbf{R}^+ X_i$ in N_c/P where i is the last (and only) element of e. It follows that the one-parameter subgroup $t \to \delta(t)$ through a, together with N, acts transitively on Ω. Thus, we may assume that N is codimension one in G, as claimed.

Actually, we can say more. The eigenvalues of $\delta(t)$ are of the form t^j for some real j. According to Proposition 24, we may assume that there is a real complement to $\mathcal{P}$ in $\mathcal{N}_c$ spanned by eigenvectors corresponding to positive j. Let $\mathcal{N}^+$ be the span of the eigenspaces of $\delta(t)$ in $\mathcal{N}$ corresponding to positive j and let $\mathcal{N}^-$ be the span of the other eigenspaces. Clearly, $\mathscr{N}^{\pm}$ are subalgebras of $\mathscr{N}$ whose direct sum is $\mathscr{N}$. Furthermore, Proposition 24 shows that $\mathscr{N}^{-} \subset \mathscr{P}$. Clearly $N = N^{+}N^{-}$.

Lemma 27. *The group AN^{+} acts transitively on Ω'.*

Proof. Consider the projection mapping $\pi: N_c/P \to N_c^{+}/P^{+}$, where $P^{+} = P \cap N_c^{+}$. Then π is an N^{+}-equivariant biholomorphism. The pair $(\mathscr{N}^{+}, \mathscr{P}^{+})$ is a Siegel $\mathscr{N}$-$\mathscr{P}$ pair in the sense of [11]. Both $(\mathscr{N}, \mathscr{P})$ and $(\mathscr{N}^{+}, \mathscr{P}^{+})$ define precisely two domains. Furthermore, the image under π of each of the $(\mathscr{N}, \mathscr{P})$ domains contains a $(\mathscr{N}^{+}, \mathscr{P}^{+})$ domain. It follows that π defines a biholomorphism of the domains, proving the lemma. q.e.d.

Thus, we may assume that $\mathscr{N}^{-} = 0$. Since the automorphism group of $\mathscr{N}$ is algebraic, we may assume that the j are rational and hence that the polarization is dilated.

Now, let $X = X_i$, where i is the last (and only) element of e. The next lemma allows us to apply some results of [11].

Lemma 28. *For all $x \in N_c$,* $\dim(N \cap xPx^{-1}) = \dim N - (2n-1)$.

Proof. The formula is equivalent to the statement that the N orbit of xP has dimension $2n-1$. Let us first consider the case $x \in \Omega'$. Then, according to (A.6) of [11],

$$xP = x_n(\exp tX)P$$

for some unique t in $\mathbf{R}^{+}$ and some $x_n \in N$. From the uniqueness of t, an element $m \exp sX$ $(m \in N)$ will fix xP only if $s = 0$. Hence, the G isotropy subgroup of xP is contained in N. Our formula follows for such points because their G orbit has real dimension $2n$ and N is codimension one in G.

Next, suppose that $x = e$. Since the identity coset is a boundary point of Ω', its G-orbit has dimension $2n-1$. On the other hand, according to (10), this point is fixed by A. This proves the lemma in this case.

Finally, to consider the general case, let d_0 be the minimal dimension of $N \cap xPx^{-1}$. The set $\mathscr{U}$ of all points x at which the dimension is d_0 is a Zariski open subset of N_c. Since Ω' is open in N_c/P, it follows that $d_0 = 2n-1$. Hence $\mathscr{U}$ is a neighborhood of e. The general result now follows from the observation that we may conjugate any element of N_c into any neighborhood of e using elements of A. q.e.d.

The significance of Lemma 28 is that it proves that Ω' is "regular" in the sense defined on p. 392 of [11]. This allows us to apply Theorem (0.3) of [11]. Actually, what we require is one of the ancillary results leading up to this result. Let $Q \subset G_c$ be A_cP. We set $n = \exp X$ and define

$B = n^{-1}Qn$. Then the subalgebra $\mathcal{Q}$ of [11, p. 406] is precisely the $\mathcal{Q}$ of the present context. The discussion beginning on p. 406 of [11] shows that there is a $\beta \in \mathcal{G}^*$ which is trivial on $\mathcal{P}+\overline{\mathcal{P}}$ such that $\mathcal{Q}$ is a polarization for β. (It is at this point that we use the nondegeneracy of the Levi form.)

The algebra $\mathcal{P}$ is not a polarization for β. In fact, since the G action is effective, there is at least one element Z which is central in $\mathcal{N}$ and which is not in $\mathcal{P}$. We may choose Z to span a one-dimensional ideal in $\mathcal{G}$. Let $\mathcal{P}' = \mathcal{P} + \mathbf{C}Z$. Then, $\mathcal{P}'$ has the same dimension as $\mathcal{Q}$ and $[\mathcal{P}', \mathcal{P}'] \subset \ker\beta$. It follows that $\mathcal{Q}$ is a polarization for β. Then it is also a polarization for $\lambda = \beta|\mathcal{N}$.

We next claim that $\mathcal{P}'$ is totally complex. For this, note that $\mathcal{P} + \overline{\mathcal{P}} = (\mathcal{Q} + \overline{\mathcal{Q}}) \cap \mathcal{N}_c$. The space $\mathcal{Q} + \overline{\mathcal{Q}}$ is codimension one in $\mathcal{G}_c$ [11, Lemma (2.7)]. Hence, it suffices to show that $Z \notin \mathcal{P} + \overline{\mathcal{P}}$. However, if $Z \in \mathcal{P} + \overline{\mathcal{P}}$, then $\lambda(Z) = 0$. But then $[\mathcal{Q}, Z] \subset \ker\beta$. This implies that $Z \in \mathcal{Q} \cap N_c = \mathcal{P}$, contradicting the choice of Z.

Knowing that $\mathcal{P}$ is the intersection of a totally complex polarization with $\ker\lambda$, it is now a simple matter to prove that Ω' is the associated nil-ball. It is also clear that $(\mathcal{N}, \mathcal{P}, \lambda)$ is dilated by the A action. This finishes the proof of the "only if" portion of Theorem 3. The converse statement is a direct consequence of Theorem 3.

3. Biholomorphisms

In this section, we study biholomorphisms between homogeneous nil-balls. As in the previous section, all groups will be simply connected, so that we shall continue the convention that upper case *Roman manuscript* letters denote Lie groups while the corresponding script letter denotes the corresponding Lie algebra. The groups in question will also be nilpotent. We shall realize the Lie group as the Lie algebra equipped with the Campbell-Hausdorff product so that the exponential map becomes the identity. Thus, for example, N and $\mathcal{N}$ represent the same space, although the former is thought of as a group and the latter as a Lie algebra.

We will assume that the notation is as in the definition of "homogeneous nil-ball" given at the beginning of §2. This gives us a Lie algebra $\mathcal{N}$ and a pair $(\lambda, \mathcal{P}')$, where λ is a linear functional and $\mathcal{P}'$ is a complex polarization for λ and a one-parameter group of automorphisms δ which leaves $\mathcal{P}'$. We assume that the action is effective, so that the kernel of λ contains no nontrivial ideals. We shall denote the corresponding domain by Ω_1. We shall need to make the realization described in §2 more explicit. First, however, we note that following well-known lemma.

Lemma 29. *Let N be a nilpotent Lie group and let M be a connected subgroup. Then there is a vector complement $\mathscr{T}$ to $\mathscr{M}$ in $\mathscr{N}$ such that the map $\phi: \mathscr{T} \times \mathscr{M} \to N$ given by $(s, t) \to st$ is a bijective polynomial mapping with polynomial inverse. If there is a subalgebra $\mathscr{M}_1$ such that $\mathscr{M}_1 + \mathscr{M} = \mathscr{N}$, then we may choose $\mathscr{T} \subset \mathscr{M}_1$.*

Proof. The first part follows from Proposition (A.3) of [11] with $K = e$. Note that in this case $\mathscr{U} = N$.

The second part will follow from the first, once it is shown that $M_1 M = N$. This again follows from [11, Proposition (A.3)], now with $K = M_1$. q.e.d.

Our realization is based upon the observation that $\overline{\mathscr{P}'} + \mathscr{P} = \mathscr{N}_c$. Thus,

$$N_c/P = \overline{P}'P/P = \overline{P}'/K_c,$$

where $K_c = \overline{P'} \cap P = \overline{P} \cap P$. Let $\mathscr{Q}$ be a δ-invariant complex complement to $\mathscr{K}_c$ in $\overline{\mathscr{P}}$ so that $\mathscr{Q} + \mathscr{Z}_c$ is a complement to $\mathscr{K}_c$ in $\overline{\mathscr{P}'}$. Then, as complex manifolds, $N_c/P \sim \mathscr{Q} \times \mathscr{Z}_c$. Since the image of N in this quotient determines the boundary of Ω, we wish to describe the image of N in $\mathscr{Q} \times \mathscr{Z}_c$. To this end, let Z_r be a basis of $\mathscr{Z}$ and let $Z_c = iZ_r$. We choose Z_r so that $\lambda(Z_r) = 1$. The function r described in the following theorem is the defining function for Ω_1.

Theorem 30. *For each $q \in \mathscr{Q}$ there is an element $p \in \mathscr{P}$ and a unique $r \in \mathbf{R}$ such that the element $g = qp(rZ_c)$ belongs to N. These elements may be chosen to depend polynomially on q. In $\mathscr{Q} \times \mathbf{C}$, the domain Ω_1 is described by* $\operatorname{im} w > r(q)$.

Proof. Let us first discuss the uniqueness of r. Suppose that $qp(rZ_c)$ and $qp'(sZ_c)$ both belong to N. Then $p^{-1}p'((s-r)Z_c)$ belongs to $P' \cap N$. If $s \neq r$, this implies that Z_c belongs to $\mathscr{P} + \mathscr{N}$. This is impossible since $\lambda(Z_c) = i$, while λ is real valued on $\mathscr{N} + \mathscr{P}$.

For the existence, note that $\mathscr{N}_c = \mathscr{N} + \mathscr{P}'$. It follows that

$$Q \subset NP' = NP(\mathbf{C}Z_r) = NP(i\mathbf{R}Z_r).$$

The existence of r follows from this, as does the polynomial dependence of r. (One applies Lemma 29, after choosing appropriate vector complements.) The fact that Ω_1 is described as claimed is clear. q.e.d.

We shall require the Levi form for Ω_1 at $(0, 0)$. This is the second-order term in the Taylor expansion of the function $r(q)$ at 0.

Corollary 31. *The Levi form is* $Z \to \frac{i}{2}\lambda([Z, \overline{Z}])$.

Proof. Let $p_1(q)$ be the order one part of the polynomial p of Theorem 30. From the Campbell-Hausdorff formula, $q + p_1(q)$ is the first-order part of $qp(q)$. This must be real modulo $\mathscr{Z}$. It follows that $p_1(q) = q + \overline{q} + k(q)$, where $k(q) \in \mathscr{K}_c$ with $\mathscr{K} = \mathscr{N} \cap \mathscr{P}$. It follows from the Campbell-Hausdorff formula that the second-order term in the Z_r component of $qp(q)$ is then

$$\lambda([q, \overline{q} + k(q)])/2 = \lambda([q, \overline{q}])/2.$$

This is a pure imaginary number. The result follows from this.

Next we consider biholomorphic mappings between non-pseudo-convex nil-balls. Let $(\mathscr{R}', \mu)$ be another effective polarization for the nilpotent Lie algebra $\mathscr{M}$. Let Ω_2 be the corresponding domain, realized in M_c/R.

Let A be a biholomorphism of Ω_1 onto Ω_2. As noted in the introduction, A extends to a biholomorphism of N_c/P onto M_c/R which restricts to an analytic diffeomorphism of the boundaries. Thus, A defines an analytic mapping A_r of $NP/P = N/K$ into $MR/R = M/L$, where $K = N \cap P$ and $L = N \cap R$. We shall assume that A maps the identity coset into the identity coset. Let B be the differential of A_r at the identity coset. This is an injective linear mapping of $\mathscr{N}/\mathscr{K}$ onto $\mathscr{M}/\mathscr{L}$.

The following seemingly innocuous proposition is actually both nontrivial and crucial. Our proof is strongly motivated by ideas in the work of Webster [14]. We note that if $\mathscr{G}$ is a Lie algebra, then $\mathscr{Z}(\mathscr{G})$ denotes the center of $\mathscr{G}$.

Proposition 32. *B maps the image of $\mathscr{Z}(\mathscr{N})$ in $\mathscr{N}/\mathscr{K}$ onto the image of $\mathscr{Z}(\mathscr{M})$ in $\mathscr{M}/\mathscr{L}$.*

Proof. We note that $K = K_c \cap N$, where $K_c = P \cap \overline{P}$. We shall refer to the space N_c/K_c as the complexification of the boundary. This space carries a canonical conjugation operator obtained as a projection from the conjugation on N_c. The boundary N/K imbeds as the set of fixed points of the conjugation operation on the complexification. The real analytic map A_r extends holomorphically to a mapping A_c of a neighborhood of $N/K = NK_c/K_c$ into M_c/L_c. As was noted above, A extends to a biholomorphism of N_c/P onto M_c/R. Consider the following diagram, where A_c is considered as a partially defined operator on N_c/K_c and where π is the obvious projection map:

$$\begin{array}{ccc} N_c/K_c & \xrightarrow{A_c} & M_c/L_c \\ \pi\downarrow & & \pi\downarrow \\ N_c/P & \xrightarrow{A} & M_c/R \end{array}$$

This diagram commutes on the real points of N_c/K_c. It follows from the uniqueness of holomorphic extension that this diagram commutes on a neighborhood of N/K in N_c/K_c.

The first conclusion we draw is that, on its domain, A_c maps left-P cosets into left-R cosets. Since A_c commutes with conjugation, it also follows that the same is true relative to $\overline{P}$ and $\overline{R}$. In particular, by restriction, we obtain the following diagram, where the vertical arrows are biholomorphisms, and, hence, A_c in fact extends holomorphically to $\overline{P}/K_c$:

$$\begin{array}{ccc} \overline{P}/K_c & \xrightarrow{A_c} & \overline{R}/L_c \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \overline{P}P/P & \xrightarrow{A} & \overline{R}R/R \end{array}$$

Note that for all k in K_c, ad $k(\mathscr{P}) = \mathscr{P}$. Thus, in general, ad $x(\mathscr{P})$ depends only on xK_c. This allows us to discuss $\operatorname{ad} x(\mathscr{P})$ for $x \in N_c/K_c$. Similarly, xP is defined for $x \in N_c/K_c$. Analogous comments hold relative to M_c/L_c and R.

Let B_c be the differential of A_c at the identity coset. Then B_c is the extension of B to $\mathscr{N}_c/\mathscr{K}_c$ by complex linearity.

Lemma 33. *Let $\tilde{q} = qK_c$, where $q \in \overline{P}$. Then*

$$B_c(\text{ad } q(\mathscr{P} + \overline{\mathscr{P}})/\mathscr{K}_c) = \text{ad } A_c(\tilde{q})(\mathscr{R} + \overline{\mathscr{R}})/\mathscr{L}_c. \tag{12}$$

Proof. Consider the variety $X = qP\overline{P}$. Then X contains $e = qeq^{-1}$. There is a neighborhood $\mathscr{V}_0$ of e in P and $\mathscr{W}_0$ of q^{-1} in $\overline{P}$ such that the image of $q\mathscr{V}_0\mathscr{W}_0$ in N_c/K_c belongs to the domain of A_c. We may also assume that the image of $q\mathscr{V}_0$ belongs to the domain of A_c. Now, A_c maps left-P cosets into left-R cosets. Thus, if $p \in P$ and both x and xp belong to the domain of A_c, then $A_c(xpK_c) \subset A_c(xK_c)R$. The same is true relative to the space $\overline{P}$ and $\overline{R}$. It follows that

$$A_c(q\mathscr{V}_0\mathscr{W}_0K_c) \subset A_c(qK_c)R\overline{R}/L_c.$$

Since A_c maps eK_c to eL_c, we conclude that B_c maps the tangent space at the identity coset of X/K_c into that of the right above. This proves that the left side of (12) is contained in the right. The equality follows by consideration of dimension, since each space has codimension one in $\mathscr{N}_c/\mathscr{K}_c$.

Now, since B is surjective, there is a vector V in $\mathscr{N}$ such that $B(V + \mathscr{K}_c)$ spans $(\mathscr{Z}(\mathscr{M}) + \mathscr{L}_c)/\mathscr{L}_c$. Proposition 32 is equivalent to the statement that $V \in \mathscr{Z}(\mathscr{N}_c) + \mathscr{K}_c$. This follows from the following lemma.

In fact, if $V \notin \mathscr{Z}(\mathscr{N}_c) + \mathscr{K}_c$, the lemma implies that there is a q in $\overline{P}$ such that $V + \mathscr{K}_c$ belongs to $\operatorname{ad} q(\mathscr{P} + \overline{\mathscr{P}})/\mathscr{K}_c$. Thus, $B_c(V + \mathscr{K}_c)$ belongs to the space on the right of (12). This space, however, intersects $\mathscr{Z}(\mathscr{M}_c)/\mathscr{L}_c$ trivially, contradicting the choice of V.

Lemma 34. *Let $V \in \mathscr{N}$. Then if V is not in $\mathscr{K}_c + \mathscr{Z}(\mathscr{N}_c)$, there is a $q \in \overline{P}$ such that $V \in \operatorname{ad} q(\mathscr{P} + \overline{\mathscr{P}})$.*

Proof. The given implies that V is not an element of $\mathscr{P}$. The Pukanszky condition for the complex polarization $\overline{\mathscr{P}}$ says that in $\mathscr{N}_c$

$$\operatorname{ad}^* \overline{P}(\lambda) = \lambda + \overline{\mathscr{P}}^{\perp}.$$

(This follows easily from the fact that the space on the left is a Zariski closed, codimension-zero subset of the space on the right; see [1].) In particular, there is a $q \in \overline{P}$ such that $\operatorname{ad}^* q(\lambda)(V) = 0$. But this implies the lemma since $\mathscr{P} + \overline{\mathscr{P}}$ is the kernel of λ. q.e.d.

Let Z_N be a basis element of $\mathscr{Z}(\mathscr{N}) \bmod \mathscr{K}$. Similarly, let Z_M be a basis element of $\mathscr{Z}(\mathscr{M})$ (mod)L. Then we have the following strengthening of the above proposition:

Corollary 35. $A_r\colon (\exp \mathbf{R}Z_N)K \to (\exp \mathbf{R}Z_M)L$.

Proof. The central Lie algebra elements Z_N and Z_M define invariant vector fields on N/K and M/L respectively. Let $\gamma_N(t) = (\exp tZ_N)K$ in N/K and let $\gamma_M(t) = A_r(\gamma_N(t))$. We claim that there is a continuous, nonzero, function $c\colon \mathbf{R} \to \mathbf{R}$ such that, for all t,

$$\gamma_M'(t) = c(t) Z_M(\gamma_M(t)).$$

Granted this, it follows from the uniqueness of integral curves that there is a function $\phi\colon \mathbf{R} \to \mathbf{R}$ such that $\gamma_M(t) = (\exp \phi(t) Z_M)_L$. Thus, the corollary will follow.

Now, for x in either N or M, we will let $L(x)$ denote left translation by x on either N/K or M/L, depending upon the context. For $t \in \mathbf{R}$, let

$$B_t = L(\sigma_M(t)^{-1}) A_r L(\exp tZ_N),$$

where $\sigma_M(t) \in M$ is such that $\sigma_M(t)M = \gamma_M(t)$. Then B_t extends to a biholomorphism of Ω which has the origin as a fixed point. From Proposition 32, there is a nonzero constant $c(t)$ such that

$$dB_t(Z_N) = c(t) Z_M \qquad \operatorname{mod} \mathscr{L}.$$

Our claim follows from the above formula after multiplying both sides by $dL(\sigma_M(t))$. q.e.d.

Let us summarize what we know concerning A. We adopt the realization of Theorem 30, except that we identify the spaces $\mathscr{Q}$ with $\mathbf{C}^n$ for both Ω_1 and Ω_2. The existence of a biholomorphism between the domains implies that the identification may be made in such a way that Ω_1 and Ω_2 have the same Levi form on $\mathbf{C}^n$. Following [2], we denote this form by $\langle Z, Z \rangle$.

Given (z, w) in $\mathbf{C}^n \times \mathbf{C}$, let $A(z, w) = (A_1(z, w), A_2(z, w))$ in $\mathbf{C}^n \times \mathbf{C}$. Then the above corollary implies that $A_2(0, w) = aw$, where a is a complex constant. (Note that $w \to A_2(0, w)$ defines a biholomorphism of $\mathbb{C}$ onto $\mathbb{C}$.) The curve $\gamma(t) = (0, t)$ in $\partial\Omega_1$ is mapped into $t \to (0, at)$. This curve must belong to $\partial\Omega_2$. It follows that a is a positive real number.

Proposition 32 says that at $(0, 0)$,

$$\frac{\partial}{\partial w} A_1 = 0.$$

Let Ω_i' be a Chern-Moser form of Ω_i. Let $\phi_i \colon \Omega_i \to \Omega_i'$ be local biholomorphisms which satisfy the Chern-Moser normalization conditions [2, p. 231]. Let $A' = (A_1', A_2')$ be the corresponding biholomorphism of the Ω_i'. Then, at $(0, 0)$,

$$\frac{\partial}{\partial w} A_1' = 0, \qquad \mathrm{re}\left(\frac{\partial^2 A_2'}{\partial w^2}\right) = 0.$$

(These conditions follow from the similar conditions for A along with the normalization conditions.) It is also true that

$$\frac{\partial^2}{\partial z^\alpha \partial z^\beta} A_2' = 0$$

at $(0, 0)$. This condition follows from the fact that A' preserves the normal forms (see the comments below formula (2.4), p. 229 of [2]).

The differential $T = dA'$ may be written in block form as

$$T = \begin{bmatrix} a & 0 \\ 0 & T_1 \end{bmatrix},$$

where T_1 is $n \times n$ and preserves the Levi form up to the factor a. (Note that T must preserve the holomorphic part of tangent space to the boundary.) We may write $A' = TA''$, where $dA'' = I$. An easy computation (cf. [2, p. 233]) shows that $T^{-1}(\Omega_2')$ is a domain in normal form and A'' is a mapping of Ω_1' into this domain. Furthermore, A'' satisfies the Chern-Moser normalization conditions [1, p. 231]. Thus, the uniqueness in Theorem (2.2) in [2] implies that $A'' = I$. This finishes the proof of Theorem 9. q.e.d.

We still need to prove Corollary 12 to finish this section, which states that any biholomorphism of the canonical forms which preserves the base point, preserves the defining function up to scalars. However, any such automorphism is a matrix transformation of the form of T above. If the boundary of Ω_2 is defined by $\operatorname{im} w - r_2(z) > 0$, then $aw - r_2 \circ T(z) > 0$ is a defining function for Ω_1. However, it follows from the implicit function theorem that defining functions of the form $\operatorname{im} w - r(z) > 0$ are unique at smooth points of the boundary. Hence $r_2 \circ T = ar_1$, proving the corollary.

4. Normal forms

Our first goal in this section is to describe an explicit defining function for a given dilated nil-ball. We shall adopt the realization of Theorem 30. We shall let $\mathscr{D}_k$ denote the t^k eigenspace of the dilation $\delta(t)$ in $\mathscr{N}_c$. We set $\mathscr{V}_j$ equal to the span of $\mathscr{D}_k$ for $k \geq j$. The $\mathscr{V}_j$ are ideals in $\mathscr{N}_c$ which satisfy $[\mathscr{N}_c, \mathscr{V}_j] \subset \mathscr{V}_{j+1}$. We shall also let d be the dilation degree of Ω. If $X \in \mathscr{N}_c$, we define $d(X) = k$, where k is the last index such that $X \in \mathscr{V}_k$. The following lemma is a simple consequence of the effectiveness of the action.

Lemma 36. *The space $\mathscr{V}_j$ is trivial if $j > d$. Furthermore, $\mathscr{V}_d = \mathscr{Z}_c$, where $\mathscr{Z}_c$ is the center of $\mathscr{N}_c$. This space is one dimensional.*

Let $\mathscr{N}^1$ be the kernel of λ in $\mathscr{N}$. Then $\mathscr{N}^1$ is, as a vector space, canonically isomorphic with $\mathscr{N}/\mathscr{Z}$. This endows $\mathscr{N}^1$ with a Lie algebra structure. Furthermore, $\mathscr{N} = \mathscr{N}^1 \times \mathscr{Z}$. In this presentation, the Lie algebra structure on $\mathscr{N}$ is given in terms of a certain two-cycle ϕ on $\mathscr{N}_1$. Explicitly, let ϕ be the bilinear form on $\mathscr{N}^1 \times \mathscr{N}^1$ defined by restricting $(X, Y) \to \lambda([X, Y])$ to $\mathscr{N}^1$. Then, the bracket on $\mathscr{N}^1 \times \mathbf{R}$ is defined by

$$[(X, s), (Y, t)] = ([X, Y], \phi(X, Y)),$$

where $[X, Y]$ is the quotient bracket on $\mathscr{N}^1$.

Note that

$$\mathscr{N}_c^1 = \sum_{i<d} \mathscr{D}_i.$$

Furthermore, $\delta(t)^* \phi = t^d \phi$. It follows that $\mathscr{D}_j$ and $\mathscr{D}_k$ are ϕ-orthogonal if $i + j \neq d$.

We may write $\mathscr{N}_c^1$ as the direct sum of the spaces $\mathscr{Q}$ and $\mathscr{P}$. We shall let π_q and π_p be the corresponding projection operators which define this decomposition. These operators commute with δ. We also define operators ρ and ν by

$$\nu(X) = \pi_q(X) + \overline{\pi_q(X)}, \qquad \rho(X) = \pi_p(X) - \overline{\pi_q(X)}.$$

Then ν and ρ are projections valued in $\mathscr{N}^1$ and $\mathscr{P}$ respectively. Furthermore, $\nu + \rho$ is the identity mapping on $\mathscr{N}_c^1$.

Let A be a polynomial mapping of $\mathscr{N}_1$ into itself. We shall say that A is scaled if A commutes with $\delta(t)$ for all t. We say that A is homogeneous of degree k if $A(tX) = t^k A(X)$ for all $t > 0$ and all X. For such A, we may write $A(X) = A'(X_1, \cdots, X_k)$, where A' is a unique, symmetric, k-linear mapping of $\mathscr{N}_1^k$ into $\mathscr{N}_1$. Then A is scaled if and only if A' commutes with the product action of $\delta(t)$ on $\mathscr{N}_1^k$. In this case, we say that A' is scaled.

Lemma 37. *Suppose that A is scaled and is homogeneous of degree k. Then A is constant on cosets of $\mathscr{V}_{d-k+1}$. In particular, if $k \geq d$, then A is the zero map. Furthermore*

$$d(A'(X_1, \cdots, X_k)) \geq \sum d(X_i) \tag{13}$$

for all $X_i \in \mathscr{N}_1$.

Proof. The first statement follows from the second. The second follows trivially from the observation that an element X has $d(X) \geq d$ if and only if $\lim_{t\to 0} t^{-d}\delta(t)X$ exists.

Now, let $q \in \mathscr{Q}$ be given. We shall define inductively an element $p \in \mathscr{P}$ such that in the N_c^1 product, $qp \in N^1$. It will be clear from the construction that p is a scaled, polynomial function of q. We begin by setting $p_1 = \overline{q} = -\rho(q)$. Then p_1 is homogeneous of degree one as a function of q. From the Campbell-Hausdorff formula it follows that

$$qp_1 = q + \overline{q} + [q, p_1]/2 + R_3,$$

where R_3 is a sum of terms homogeneous of degree three or more. Since $q + \overline{q}$ is real, we say that this expression is real up to second degree. Let $h_2 = -\rho([q, p_1]/2)$ and let $p_2 = p_1 + h_2$. Then

$$qp_2 = q + p_1 + h_2 + [q, p_1]/2 + [q, h_2]/2 + R_3,$$

where R_3 is a sum of terms homogeneous of degree three or more (probably different from the previous R_3). By construction, $h_2 + [q, p_1]/2$ is real. On the other hand, $[q, h_2]$ defines a third degree polynomial function. Thus, qp_2 is real to third degree.

Now, suppose that operators h_i, $i = 2, \cdots, n$, have been defined so that each h_i is ith degree, scaled, and qp_n is real up to the $(n + 1)$st degree, where

$$p_n = p_1 + h_2(q) + \cdots + h_n(q). \tag{14}$$

Let $c_{n+1}(q)$ be the $(n+1)$st degree term in $qp_n(q)$, i.e., $q \to qp_n(q) - c_{n+1}(q)$ is of degree $n+2$. Let

$$h_{n+1}(q) = -\rho(c_{n+1}(q)). \tag{15}$$

Let $p_{n+1} = p_n + h_{n+1}$. Then qp_{n+1} is real up to order $n+2$. Let p denote p_n, where n is any integer greater than the dilation degree. Clearly, the value of p is independent of the specific n. This constructs an element p in $\mathscr{P}'$ such that qp is real modulo $\mathscr{Z}_c$. To construct r, one forms the product qp in $\mathscr{N}_c$, where $\mathscr{N}_c$ is identified with $\mathscr{N}_c^1 \times \mathbf{C}$. Then r is the negative of the imaginary component of the $\mathbf{C}$ coordinate of this product. Explicitly,

$$\begin{aligned} r(q) = &-\operatorname{im}(\phi(q, p)/2 + \phi(q, [q, p])/12 \\ &+ \phi(p, [p, q])/12 + \phi(q, [p, [q, p]])/24 + \cdots). \end{aligned} \tag{16}$$

Next, we turn to the transformation to normal form. Let $r = \sum_k r_k$, where r_k is homogeneous of degree k. We choose a basis Z_i, $i = 1, \cdots, n$, of $\mathscr{Q}$ over $\mathbf{C}$ consisting of eigenvectors of $\delta(t)$. We define a basis for $\mathscr{Q}$ over $\mathbf{R}$ by setting $Q_i = Z_i$ for $1 \le i \le n$, and $Q_{i+n} = J(Z_i)$ for $i = n+1, \cdots, 2n$, where J is multiplication by i on $\mathscr{Q}$. We use this basis to identify $\mathscr{Q}$ with $\mathbf{R}^{2n}$. Following Chern-Moser [2, p. 232] and using the notation of tensor calculus, we write

$$r_k(x) = \sum a_{\alpha_1 \cdots \alpha_k} x^{\alpha_1} \cdots x^{\alpha_k}.$$

We shall say that the indices i and j are separated if $d(Q_i) + d(Q_j) \ge d$.

Lemma 38. *Suppose that $k > 2$. Then, in the above summation, the term corresponding to $\alpha_1, \cdots, \alpha_k$ will be zero if any two indices from the sequence $\alpha_1, \cdots, \alpha_k$ are separated.*

Proof. Any nth degree, homogeneous, polynomial function on a vector space $\mathscr{V}$ may be expressed as a symmetric n-form evaluated on the diagonal in $\mathscr{V}^n$. Our lemma is equivalent to the statement that the form corresponding to r_k is zero whenever any pair of its arguments equal (Q_i, Q_j), where i and j are separated.

However, from (16), the tensor corresponding to r_k is expressible as a linear combination of symmetrizations of terms of the form

$$\phi(X_1, A(X_2, \cdots, X_k)),$$

where A is a scaled operator of degree $k-1$. Our result follows from (13) and the observation that if $d(X) + d(Y) > d$, then $\phi(X, Y) = 0$. q.e.d.

We may write

$$r_k(q) = \sum_{i+j=k} r_{i,j}(q, \overline{q}),$$

where $r_{i,j}(\alpha q, \beta\overline{q}) = \alpha^i\beta^j r_{i,j}(q, \overline{q})$ for all complex α and β. Since r_k is real $r_{i,j} = \overline{r}_{j,i}$. In [2, p. 232], certain polynomials, called traces, are defined from the $r_{i,j}$. We claim that in our case, these traces are automatically zero. Granted this, in the notation of [2, formula (2.11)], the only further condition required to bring Ω into normal form is $N_{k,l} = 0$ for $\min(k, l) < 1$. We shall give an inductive definition of a transformation of Ω which achieves this condition while retaining the trace zero condition. First, however, we prove our claim.

Lemma 39. *Suppose that* $i \geq 1$, $j \geq 1$, *and* $i+j \geq 2$. *Then* $\operatorname{tr}(r_{i,j}) = 0$.

Proof. From the proof of (16), $r_2(q) = -\operatorname{im}\phi(q, \overline{q})/2$. Thus, the Levi form for Ω is

$$H(z, w) = \frac{i}{2}\phi(z, \overline{w}).$$

Let $g_{i,j} = H(Z_i, Z_j)$. Then $g_{i,j} = 0$ if $d(Z_i) + d(Z_j) \neq d$. The same is true for the inverse $g^{i,j}$. Our claim now follows from the observation that, from formula (2.10) of [2], the coefficient of each term used in computing the traces has at least two separated indices. q.e.d.

Now we shall describe the transformation into normal form. Note that $r_{k-1,1}$ is linear in $\overline{q}$. Thus there is a function f_{k-1} such that

$$r_{k-1,1}(q, \overline{q}) = H(f_{k-1}(q), q).$$

Then f_{k-1} is a holomorphic, polynomial mapping of $\mathcal{Q}$ into itself which is homogeneous of degree $k-1$ over $\mathbf{C}$. It also follows that f_{k-1} is scaled. This is due to the fact that both r and H are homogeneous of weight d with respect to δ.

Now, let

$$f(q) = q + \sum_{i\geq 2} f_i(q).$$

Let $r_{\infty,1}$ be the sum over k of the terms $r_{k,1}$. Then

$$r_{\infty,1}(q, \overline{q}) = H(f(q), q). \tag{17}$$

Lemma 40. *There is a unique scaled polynomial mapping* $h\colon \mathcal{Q} \to \mathcal{Q}$ *without terms homogeneous of degree zero or one, such that*

$$f \circ (I + h) = I - h,$$

where I *is the identity mapping on* $\mathcal{Q}$.

Proof. The above equality is equivalent with

$$f_2 \circ (I + h) + \cdots + f_n \circ (I + h) = -2h.$$

Equating the degree k homogeneous parts of the left- and right-hand sides of this equality for $k = 1, \cdots, n$ shows that h may be computed inductively. q.e.d.

Let $g = I + h$ and let $\tilde{r} \circ g$. Let $\tilde{\Omega}$ be defined as the set of points in $\mathscr{Q} \times \mathbf{C}$ such that $\operatorname{im} w > \tilde{r}(q)$.

Theorem 41. *The domain $\tilde{\Omega}$ is the Chern-Moser normal form of the domain associated with the pair $(\mathscr{N}, \mathscr{P})$.*

Proof. One of the defining conditions for the normal form is that there exist no terms in the defining function of the form of $\tilde{r}_{k,1}$ or $\tilde{r}_{1,k}$ for $k > 0$. The other defining conditions are trace conditions. These latter conditions follow essentially from Lemma 38 since composition with g will not introduce any nonseparated indices in the trace computations. (Note that from (13), $h: \mathscr{V}_k \to \mathscr{V}_{k-1}$ for all k.) Thus, we need only consider the former conditions.

The term $\tilde{r}_{\infty,1}$ is (up to a choice of sign) that part of $r_{\infty,1} \circ g + r_{1,\infty} \circ g$ which is complex linear in q. Explicitly, from (17), this is the linear part of

$$H(I - h, I + h) + H(I + h, I - h) = 2(H(I, I) - H(h, h)),$$

which is just the Levi form, as desired. q.e.d.

The above computations have an immediate and interesting consequence. Note that the normal form, by definition, contains no terms of degree three. On the other hand, if the dilation degree of the domain is 3 or less, than the normal form will also contain no terms of degree 4 or greater. Our conclusion is:

Corollary 42. *If the dilation degree of Ω is 3 or less, then Ω is biholomorphic to either the domain $\operatorname{im} w > H(z, z)$ or $\operatorname{im} w > -H(z, z)$. In particular, the isomorphism class of $\partial\Omega$ is determined by the signature of the Levi form.*

In degree 4, we can also be quite explicit. We write $\mathscr{N}_1 = \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3$, where the $\mathscr{D}_1$ are as before. Then space $\mathscr{Q}$ decomposes accordingly to the sum of complex subspaces $\mathscr{Q}_i$. From homogeneity, $\mathscr{Q}_i$ and $\mathscr{Q}_j$ are H-orthogonal if $i + j \neq 4$. For $q \in \mathscr{Q}$, we write $q = q_1 + q_2 + q_3$ in this decomposition. From the above considerations, the degree 4 terms in r are a sum of terms of the form $\phi(u(q), v(q))$, where u and v are scaled functions whose degrees sum to 4. From the proof of Lemma 37, this function depends only upon q_1. Furthermore, it is clear that there are no terms of higher degree than 4. Thus, considering the orthogonality of the $\mathscr{Q}_i$, we see that

$$r(q) = \operatorname{re}(2H(q_1, q_3) + H(q_2, q_2) + Q(q_1)),$$

where Q is some fourth degree polynomial. This is clearly of the form claimed in Corollary 11. The proof of Corollary 11 will be complete once we have proved the following:

Lemma 43. *The domain defined by* (3) *is homogeneous of type* B.

Proof. Let $a \in \mathbf{C}^n$. Consider the real, degree 3 polynomial $R^a(z) = Q(z+a) - Q(z)$. We may write

$$R^a(z) = \operatorname{re}(R_0^a(z) + R_1^a(z, \overline{z})),$$

where R_0^a is holomorphic and R_1^a is linear in $\overline{z}$ and holomorphic in z. There is a holomorphic polynomial function f_a mapping $\mathbf{C}^n$ into itself such that

$$R_1^a(z, \overline{z}) = f_a(z) \cdot \overline{z}.$$

Define a transformation T_a mapping $\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k \times \mathbf{C}^n$ by $T_a(\tau, z, \rho, w) = (\tau', z', \rho', w')$, where $z' = z + a$, $\rho' = \rho$, $w' = w - f_a(z)$, and $\tau' = \tau + i(R_0^a(z) - \overline{a} f_a(z) + \overline{a} w)$. It is clear that

$$\operatorname{im} \tau' - \operatorname{re} w' \overline{z}' - Q(z') = \operatorname{im} \tau - \operatorname{re} w \overline{z} - Q(z).$$

We similarly define a transformation S_a by $z' = z$, $\rho' = \rho$, $w' = w + a$, and $\tau' = \tau + i\overline{a}z$. For each $b \in \mathbf{C}^k$, we define U_b by $\rho' = \rho + b$ and $\tau' = \tau + i(2H(\rho, b) + H(b, b))$. (The other variables are unchanged.)

These transformations all leave the domain defined by (3) invariant. There is also a "dilation" of this domain defined by

$$\delta(t)(\tau, z, \rho, w) = (t^4\tau, tz, t^2\rho, t^3 w).$$

Translation of τ in the imaginary direction also leaves the domain invariant. Clearly, the group generated by the above transformations acts transitively on Ω, making Ω into a type B domain.

5. Holomorphically abelian domains

Let $(\mathscr{N}, \mathscr{P}', \lambda, \delta)$ define an effective, dilated nil-ball as in the previous section. In this section, we make the additional assumption that $\mathscr{P} = \mathscr{P}' \cap \ker \lambda$ is abelian. It follows that $\overline{\mathscr{P}} \cap \mathscr{P} = 0$, since this space is an ideal in the kernel of λ. We say that the given data defines a *holomorphically abelian* domain. We set $\mathscr{N}^1 = \mathscr{N}/\mathscr{Z}(\mathscr{N})$. Note that $\mathscr{N}^1$ is a Lie algebra. The projection of $\mathscr{N}_c$ onto $\mathscr{N}_c^1$ is injective on $\mathscr{P}$. We identity $\mathscr{P}$ with its image in $\mathscr{N}_c^1$. Then $\mathscr{P}$ defines a complex structure on $\mathscr{N}^1$. Explicitly, there is a real linear mapping J on $\mathscr{N}^1$ with $J^2 = -I$ such that $\mathscr{P}$ is the $-i$ eigenspace of J. The following lemma follows easily from the assumption that $\mathscr{P}$ is abelian.

Lemma 44. *For all X and Y in $\mathscr{N}^1$, $[JX, JY] = [X, Y]$.*

Consider the bilinear form ϕ_0 on $\mathscr{N}$ defined by

$$\phi_0(X, Y) = \lambda([X, Y]).$$

It is easily seen that the kernel of ϕ_0 is exactly $\mathscr{Z}(\mathscr{N})$. Let ϕ be the non-degenerate form obtained from projecting ϕ_0 to $\mathscr{N}^1$. This form satisfies the important cocycle identity

$$\phi([X, Y], Z) = \phi([X, Z], Y) + \phi(X, [Y, Z]).$$

It is also easily seen that ϕ is J-invariant. We define a Hermitian form H by

$$H(X, Y) = \phi(JX, Y) + i\phi(X, Y).$$

We define a binary operation "$*$" on $\mathscr{N}^1$ by:

$$\phi(X * Y, Z) = -\phi(Y, [X, Z]).$$

Lemma 45. *For all X and Y in $\mathscr{N}^1$, $[X, Y] = X * Y - Y * X$. Furthermore,*

$$\phi(X * X, Z) = \phi(Y, Z * X)$$

for all Z in $\mathscr{N}^1$.

Proof. One easily verifies that

$$\phi(X * Y - Y * X, Z) = \phi([X, Y], Z)$$

for all $Z \in \mathscr{N}^1$, proving the first claim. As to the second, let $L(X)$ and $R(X)$ denote, respectively, left and right multiplication by X with respect to $*$. Then $L(X) - R(X) = \mathrm{Ad}(X)$, as was just shown. On the other hand, by definition, $L(X) = -(\mathrm{Ad}\, X)^\phi$, where the superscript denotes adjoint with respect to ϕ. Therefore, $R(X) = L(X) + L(X)^\phi$, proving that $R(X) = R(X)^\phi$, as claimed. q.e.d.

Next we define two additional products on $\mathscr{N}^1$. Let

$$XY = (X * Y - JX * JY)/2, \tag{18}$$

$$X \circ Y = (X * Y + JX * JY)/2. \tag{19}$$

It is easily verified from the definition that $*$ is complex linear in the first argument. It follows that the first of the above products is complex linear in each variable, while the second is complex linear in X and antilinear in Y. Note that $X*Y = X\circ Y + XY$. Thus, the two products above determine the Lie structure on $\mathscr{N}^1$. Actually, either one of these products, together with the form H, is sufficient to determine the Lie Structure. This, and much more, is contained in the following theorem which is one half of Theorem 13.

Theorem 46. *The operation* $(X, Y) \to XY$ *makes* $\mathcal{N}^1$ *into a duality algebra relative to the form* H. *The projection of* δ *to* $\mathcal{N}^1$ *makes* $\mathcal{N}^1$ *into a dilated duality algebra. The Siegel pair defined from this duality algebra is isomorphic with* $(\mathcal{N}, \mathcal{P})$.

Proof. The fact that $XY - YX = 0$ is clear from (18) and Lemma 44. It follows that $[X, Y] = X \circ Y - Y \circ X$ for all X and Y. It also follows from Lemma 45 that

$$\phi(XY, Z) = \phi(X, Z \circ Y).$$

Using complex linearity and antilinearity, it is easily seen that this formula holds with H in place of ϕ.

We need to prove (6) for "$\circ$." This will follow from the Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]. \tag{20}$$

We expand this using "$\circ$." This left side expands as

$$X \circ (Y \circ Z - Z \circ Y) - (Y \circ Z - Z \circ Y) \circ X.$$

Among the four trinomials appearing in this expansion, only $X \circ (Y \circ Z)$ is linear in both X and Z and antilinear in Y. Expanding the right side, we find exactly one with the same linearity properties: $Z \circ (Y \circ X)$. These two terms must be equal, proving (6). We also note that the left side of (20) contains no terms linear in X and antilinear in both Y and Z. Comparison with the right side results in the identity $(X \circ Y) \circ Z = (X \circ Z) \circ Y$. Dualizing results in the identity, $(WZ)Y = (WY)Z$, which implies associativity.

The remainder of the theorem is easily shown. q.e.d.

To finish the proof of Theorem 13, we need only consider the converse statement. Thus, suppose that $(\mathcal{A}, H, \delta)$ defines a dilated duality algebra. The main point is to show that the bracket operation $[X, Y] = X \circ Y - Y \circ X$ satisfies the Jacobi identity. This, however, is straightforward from (6) and the following identity, which is a consequence of (4):

$$(Z \circ X) \circ Y = Z \circ (XY) = (Z \circ Y) \circ X.$$

The rest of the converse is simple and is left to the reader as an exercise.

Now we turn to the proof of Theorem 14. Our first step in the proof will be to implement the algorithm of §4 for computing the function r of Theorem 30.

Let J denote the complex multiplication on $\mathscr{A}$. We shall consider $\mathscr{A}$ as an algebra over $\mathbf{R}$. Then $\mathscr{A}_c$ will denote the complexification of $\mathscr{A}$. We extend the various binary operations on $\mathscr{A}$ to $\mathscr{A}_c$ by complex bilinearity. Recalling that $\mathscr{P}$ is the $+i$ eigenspace of J, it is easily seen that $\mathscr{P}$ is a subalgebra of $\mathscr{A}_c$. Furthermore we have the following containments which follow from the Hermitian linearity of "$\circ$" on $\mathscr{A}$:

$$\mathscr{P}\circ\mathscr{A}_c\subset\mathscr{P}, \quad \mathscr{Q}\circ\mathscr{A}_c\subset\mathscr{Q}, \quad \mathscr{A}_c\circ\mathscr{P}\subset\mathscr{Q}, \quad \mathscr{A}_c\circ\mathscr{Q}\subset\mathscr{P}.$$

In particular, if $q\in\mathscr{Q}$, then

$$p_2=\overline{q}-\rho(q\circ\overline{q}-\overline{q}\circ q)/2=\overline{q}+\overline{q}\circ q.$$

To compute p_3, we note that from (15), $p_3=p_2-\rho(c_3(q))$, where $c_3(q)$ is the third degree term of the Campbell-Hausdorff product of qp_2. This term is

$$c_3(q)=[q,\overline{q}\circ q]/2+[q,[q,\overline{q}]]/12+[\overline{q},[\overline{q},q]]/12.$$

Since ρ is (in this case) zero on real terms, only the first term contributes to p_3. Thus,

$$p_3=\overline{q}+\overline{q}\circ((\overline{q}\circ q)\circ q+\overline{q}\circ(q\circ\overline{q}))/2.$$

According to Theorem 13, $r(q)$ is the negative of the imaginary part of the $\mathscr{Z}$ component of the Campbell-Hausdorff product of qp_3 in $\mathscr{N}_c$. Explicitly, this component is given by (16):

$$r(q)=-\operatorname{im}(\phi(q,p)/2+\phi(q,[q,p])/12+\phi(p,[p,q])/12 \\ +\phi(q,[p,[q,p]])/24).$$

We make use of several observations to simplify this. First, we note that it is only the imaginary part of the above expression that enters into the definition of r. Second, we have the identities of Lemma 45 and formulas (4), and (5) at our disposal. Finally, any term in this expression of degree five or greater in q will be trivial. After some patient computing, we obtain:

$$r(q)=-\tfrac{1}{2}\operatorname{im}(\phi(q+q^2+q^3/3,\overline{q})-\tfrac{3}{4}\phi(\overline{q}\circ q,q\circ\overline{q}))+\cdots,$$

where the powers represent products in the associative algebra structure of $\mathscr{A}$ and the dots represents terms of degree 5 or greater.

Next, we transform our domain into canonical form. According to the proof of Theorem 41, the defining function of the normal form is the function $r'=r\circ g$, where $g=I+h$ and $f\circ(I+h)=I-h$. Furthermore, from the above arguments

$$f(q)=q+q^2+q^3/3+\cdots,$$

where the omitted terms are of degree 4 or greater. From the proof of Theorem 41, we need only know the first three terms of f to compute the corresponding terms of g. Explicitly, following the algorithm of Theorem 41, we see that

$$g(q) = q - q^2/2 + q^3/3 + \cdots .$$

Composing with r, simplifying, and ignoring real terms we obtain:

$$r'(q) = \frac{i}{8}(4\phi(q, \overline{q}) + \phi(q^2, \overline{q}^2) - 3\phi(\overline{q} \circ q, q \circ \overline{q})).$$

This still is not the normal form claimed in Theorem 14. The final simplification is to compose with the map $x \to x - iJx$, which maps $\mathscr{A}$ onto $\mathscr{Q}$. This yields the normal form of Theorem 14. This finishes the proof of this theorem. q.e.d.

Finally, we turn to the proof of Theorem 15. Thus, we assume that the dilation degree of $\mathscr{A}$ is 3 (so that $\mathscr{N}$ is 4 step.) In this case we write

$$\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3, \tag{21}$$

where the $\mathscr{A}_i$ are the $t^{n(i)}$ eigenspaces of δ, listed according to increasing $n(i)$. Let d be the degree of homogeneity of H. Clearly, $\mathscr{A}_i$ will be orthogonal to $\mathscr{A}_j$ if $n(i) + n(j) \neq d$. Thus, we see that $n(1) = n(3) = d = 2n(2)$. We consider the linear endomorphism $\delta(t)'$ of $\mathscr{A}$ defined by requiring that the $\mathscr{A}_i$ be the t^i eigenspaces of $\delta(t)'$. Then $(\mathscr{A}, \delta', H)$ is a dilated duality algebra which defines the same domain as our original triple. Hence we may assume that $\delta' = \delta$ and hence that $d = 4$, and $n(i) = i$.

The form H identifies $\mathscr{A}_3$ with the conjugate dual space of $\mathscr{A}_1$. Thus, if we choose a particular linear isomorphism of $\mathscr{A}_1$ with $\mathbf{C}^n$, then we may identify $\mathscr{A}_3$ with $\mathbf{C}^n$ in such a way that for z in $\mathscr{A}_1$ and w in $\mathscr{A}_3$,

$$H(z, w) = (z, w)/2,$$

where $(\cdot, \cdot)$ is the standard Hermitian scalar product. Since $\mathscr{A}_2$ is H-orthogonal to the other $\mathscr{A}_i$, it follows that after these identifications, H is uniquely determined by its restriction to $\mathscr{A}_2$. Let s be the signature of this restriction. We may identify $\mathscr{A}_2$ with $\mathbf{C}^n$ in such a way that this restriction becomes the form H_s of (3).

Now let $q \in \mathscr{A}$. Let $q = z + \rho + w$ be the decomposition defined by (21). Then, from Theorem 14,

$$r(q) = \operatorname{re}(z, w) + H_s(\rho, \rho) - Q(z),$$

where

$$Q(s) = H_s(z^2, z^2) + 3H_s(z \circ z, z \circ z). \tag{22}$$

This is clearly of the form claimed in Theorem 15. To prove the converse statement, let $\mathscr{E} = \mathbf{C}^n \otimes \mathbf{C}^n$. Let $C = [c_{\alpha,\beta}]$ be the matrix discussed above the statement of Theorem 15. Then C is the matrix of a Hermitian form H_C on $\mathscr{E}$ whose signature is the Hermitian signature of C. This form expresses Q as

$$Q(z) = 3H_C(z \otimes z, z \otimes z).$$

Now, assume that s dominates (p, q) as in Theorem 15. This implies that there is a linear transformation B from $\mathscr{E}$ into $\mathbf{C}^k$ such that, for all z and w in $\mathscr{E}$,

$$H_C(z, w) = H_s(Bz, Bw).$$

The symmetry properties of C imply that we may choose B such that $B(z \otimes w) = B(w \otimes z)$ for all z in $\mathbf{C}^n$.

We use B to define a product on $\mathscr{A} = \mathbf{C}^n \times \mathbf{C}^k \times \mathbf{C}^n$ as follows:

$$(z, \rho, w)(z', \rho', w') = (0, B(z \otimes z'), 0).$$

This makes $\mathscr{A}$ into an abelian, associative algebra over $\mathbf{C}$. We equip $\mathscr{A}$ with the Hermitian product H defined by

$$H((z, \rho, w), (z', \rho', w')) = ((z, w') + (w, z'))/2 + H_s(\rho, \rho').$$

We define a dilation $\delta(t)$ by declaring that $\delta(t)(z, \rho, w) = (tz, t^2\rho, t^3 w)$. It is now a straightforward verification that $(\mathscr{A}, H, \delta)$ is a duality and that the domain it defines is given by (3). In this case, $\mathscr{A}_1 \circ \mathscr{A}_1 = \{0\}$ of (3) has no terms involving $q \circ \overline{q}$.

To finish the proof of Theorem 15, we must consider the case when s dominates the anti-Hermitian index of Q. In this case, it will turn out that the algebra structure on $\mathscr{A}$ is such that $\mathscr{A}_1^2 = \{0\}$ but $\mathscr{A}_1 \circ \mathscr{A}_1$ is not zero. The proof is more or less the same as in the previous case, with a few modifications. In this case, we produce a linear transformation B of $\mathscr{E}$ into $\mathbf{C}^k$ such that

$$Q(z) = H_s(B(z \otimes \overline{z}), B(z \otimes \overline{z})).$$

We define a bilinear mapping of $\mathbf{C}^k \times \mathbf{C}^n$ into $\mathbf{C}^n$ (which will be denoted by juxtaposition) by requiring that

$$H_s(B(z \otimes \overline{w}), v) = (z, vw)$$

for all z and w in $\mathbf{C}^n$ and all v in $\mathbf{C}^k$. We then define $\mathscr{A}$, δ, and H as before, except that the algebra structure of $\mathscr{A}$ is now defined by

$$(z, \rho, w)(z', \rho', w') = (0, 0, z\rho' + z'\rho).$$

One easily verifies that we again obtain a duality algebra which defines the domain of (3). This finishes the proof of Theorem 15.

Example. *A domain with a solvable automorphism group.* It follows from Theorem 15 that the following domain is a holomorphically abelian domain in $\mathbf{C}^4$:

$$\operatorname{im} \tau > \operatorname{im} z\overline{w} + |q|^2 - |z|^4 .$$

We shall compute the full automorphism group of this domain. Done in full detail, the computation becomes somewhat long and tedious. To avoid this, we shall merely summarize the results of the computations. From Corollary 12, it follows that the group of base preserving automorphisms is generated by the dilation, together with the linear mappings on (τ, z, ρ, w)-space defined by the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & a & e^{i\psi} & 0 \\ 0 & r - i|a|^2 & -2i\overline{a} & e^{i\theta} \end{bmatrix} .$$

Let T be the group generated by the A above and $\delta(t)$. The full automorphism group is NT, where N is the nilpotent group defined by the corresponding duality algebra $\mathscr{A}$. Explicitly, $\mathscr{A}$ is $\mathbf{C}^3$ with the product defined by

$$(z, \rho, w)(z', \rho', w') = (0, zz', 0) .$$

The Hermitian form is

$$H((z, \rho, w), (z', \rho', w')) = (\overline{z}'w - z\overline{w}')/2i + \rho\overline{\rho}' .$$

This yields the duality product

$$(z, \rho, w) \circ (z', \rho', w') = (0, 0, 2i\rho\overline{a}) .$$

The transformation into normal form is given by $q \to q + q^2/2$, where $q \in \mathscr{A}$ (see Lemma 40). After (1) computing the N action on Ω and (2) mapping into normal form, we see that in normal form, the typical element $a \in \mathscr{A}$ acts on the domain by

$$a(\tau, q) = (\tau + \tau_0, q + a - a \circ a/2 + a^2/2 - (q - q^2/2) \circ a + qa)) ,$$

where τ_0 depends only on a and q. Using this formula and the formula for A given above, it is easy to compute products, commutators, etc. It is easily seen that the automorphism group is indeed solvable. (This certainly is indicated by the form by A.)

We close this section with another class of holomorphically abelian domains which demonstrates further the considerable variety possible for such domains. Let $\mathscr{A}$ be any commutative, associative, nilpotent algebra over $\mathbf{R}$ (e.g., the free-nilpotent, abelian algebra of nilpotent degree n on k generators). Let $\lambda \in \mathscr{A}^*$ be any nonzero linear functional. Let B_λ be the belinear form on $\mathscr{A}$ defined by

$$B_\lambda(X, Y) = \lambda(XY).$$

Let $\mathscr{I}$ be the radical of this form. Let $\mathscr{B} = \mathscr{A}/\mathscr{I}$. Then B_λ projects to a nondegenerate form B on $\mathscr{B}$ with the property that

$$B(XY, Z) = B(X, YZ)$$

for all X, Y, and Z. let $\mathscr{B}_c$ be the complexification of $\mathscr{B}$. We may extend B uniquely to a Hermitian form H on $\mathscr{B}_c$. The pair $(\mathscr{B}_c, H)$ is a duality algebra. The "0" operation is obtained by extending the associative product on $\mathscr{B}$ to a Hermitian-bilinear binary operator on $\mathscr{B}_c$. Several examples of such domains are worked out in [10]. They can have arbitrarily large nilpotent degree.

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